BEE003 Advanced Control System
Unit -I

PART –A (2 MARKS)

1. **Why is negative feedback preferred in a control systems?**
The negative feedback results in better stability in steady state and rejects any disturbance signals. It’s also has low sensitivity to parameter variations. Hence negative feedback is preferred in closed loop systems.

2. **What is time variant and time invariant system?**
A system is said to be time invariant if its input output characteristics do not change with time. A linear time invariant system can be represented by constant coefficient differential equations.

3. **What is transfer function?**
The Transfer function of a system is defined as the ratio of the Laplace transform of output to Laplace transform of input with zero initial conditions. It’s also defined as the Laplace Transform of the impulse response of system with zero initial conditions.

4. **What is centroid? How the centroid is calculated ?**
The meeting point of the asymptotes with the real axis is called centroid.
The centroid is given by
Centroid = (sum of poles – sum of zeros) / (n-m)

5. **What are break away and break in points?**
At break away point the root locus breaks from the real axis to enter into the complex plane. At break in point the root locus enters the real axis from the complex plane. To find the break away or break in points, form a equation for K from the characteristic equation and differentiate the equation of K with respect to s. Then find the roots of the equation dK/dS = 0. The roots of dK/dS = 0 are break away or break in points provided for this value of root the gain K should be positive and real.

6. **What do you mean by pole and zero of a transfer function?**
The pole of a function F(S) is the value at which the function F(S) becomes infinite.
The zero of a function F(S) is the value at which the function F(S) becomes zero.

7. **What is Nyquist stability criterion?**
   If the Nyquist plot of the open loop transfer function G(s) corresponding to the nyquist contour in the S- plane encircles the critical point –1+j0 in the contour in clockwise direction as many times as the number of right half S-plane poles of G(s),the closed loop system is stable.

8. **Define bandwidth of a system.**
   The bandwidth is the range of frequencies for which the system gain is more than -3 db. The bandwidth is a measure of the ability of a feedback system to reproduce the input signal, noise rejection characteristics and rise time.

9. **What is frequency response?**
   A frequency response is a steady state output of the system, when the input to the system is a sinusoidal signal.

10. **What is resonant Peak (μ_r)?**
    The maximum value of the magnitude of closed loop transfer function is called resonant peak.

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Unit -I
PART –B (6 MARKS)

1. **Derive the transfer function from state model.**
   Let

The state model of a linear time invariant system is given by

$$\dot{X}(t) = A\,X(t) + B\,U(t)$$

$$Y(t) = C\,X(t) + D\,U(t)$$

Obtain the expression for transfer function of the system.

Solution : Given that $\dot{X}(t) = A\,X(t) + B\,U(t)$ ...

and $Y(t) = C\,X(t) + D\,U(t)$ ...

On taking laplace transform of equ(1) with zero initial conditions we get,

$$s\,X(s) = A\,X(s) + B\,U(s)$$

$$s\,X(s) - A\,X(s) = B\,U(s)$$

$$(sI - A)\,X(s) = B\,U(s) \quad ....$$

On premultiplying equ(3) by $(sI-A)^{-1}$ we get

$$X(s) = (sI - A)^{-1}\,B\,U(s)$$

On taking laplace transform of equ(2) we get,

$$Y(s) = C\,X(s) + D\,U(s)$$

Substitute for X(s) from equ(4) in equ(5) we get,

$$Y(s) = C\,[(sI - A)^{-1}\,B\,U(s)] + D\,U(s)$$

$$= [C\,(sI - A)^{-1}\,B\,U(s) + D]\,U(s)$$

$$\therefore \frac{Y(s)}{U(s)} = C\,(sI - A)^{-1}\,B\,U(s) + D$$

The equation (6) is the transfer function of the system.

2. Derive the frequency domain specification bandwidth.

BANDWIDTH (ω_b)

Let, Normalized bandwidth, $u_b = \frac{\omega_b}{\omega_n}$

When $u = u_b$, the magnitude M, of the closed loop system is $1/\sqrt{2}$ (or −3db).

Hence in the equation for M (equation 4.9), put $u = u_b$ and equate to $1/\sqrt{2}$.

$$\therefore M = \frac{1}{\left[(1-u_b^2)^2 + 4\zeta^2 u_b^2\right]^{\frac{1}{2}}} = \frac{1}{\sqrt{2}}$$

On squaring and cross multiplying we get,

$$(1-u_b^2)^2 + 4\zeta^2 u_b^2 = 2 \quad \Rightarrow \quad 1 + u_b^4 - 2u_b^2 + 4\zeta^2 u_b^2 = 2 \quad \Rightarrow \quad u_b^4 - 2u_b^2(1-\zeta^2) - 1 = 0$$

Let $x = u_b^2$; $\therefore x^2 - 2(1-2\zeta^2)x - 1 = 0$

$$\therefore x = \frac{2(1-2\zeta^2) \pm \sqrt{4(1-2\zeta^2)^2 + 4}}{2} = \frac{2(1-2\zeta^2) \pm 2\sqrt{(1+4\zeta^4 - 4\zeta^2) + 1}}{2}$$

Let us take only the positive sign,

$$\therefore x = 1 - 2\zeta^2 + \sqrt{2 - 4\zeta^2 + 4\zeta^4}$$

But, $u_b = \sqrt{x}$; $\therefore u_b = \sqrt{x} = \left[1 - 2\zeta^2 + \sqrt{2 - 4\zeta^2 + 4\zeta^4}\right]^{\frac{1}{2}}$; Also, $u_b = \frac{\omega_b}{\omega_n}$

$$\therefore \text{Bandwidth, } \omega_b = \omega_n u_b = \omega_n \left[1 - 2\zeta^2 + \sqrt{2 - 4\zeta^2 + 4\zeta^4}\right]^{\frac{1}{2}}$$

3. What is the effect of addition of poles & zeros on root locus.

The addition of a pole to the open-loop transfer function has the effect of pulling the root locus to the right, which reduce the relative stability of the system and increase the settling time.

The addition of a zero to open to open loop transfer function will pull the root locus to the left which make the system more stable and reduce the settling time.

4. What are the properties of transfer function and its limitations?

<u>Properties Of Transfer Function.</u>

- Zero initial condition
- It is same as Laplace Transform of its Impulse Response
- Replacing "s" by d/dt in the Transfer Function, the differential equation can be obtained.

- Poles and Zeros can be obtained from the Transfer Function.
- Stability can be known.
- Can be applicable to Linear System only.

Limitations.

1. Transfer function is defined under zero initial conditions.
2. Transfer function is applicable to linear time invariant systems.
3. Transfer function analysis is restricted to single input and output systems.
4. Does not provides information regarding the internal state of the system.

5. Discuss the correlation between frequency response and time response.

(i) Peak response M_p in frequency domain is a function of only ζ and it is independent of ω_n.

(ii) Maximum overshoot M_p in time domain is also a function of only ζ and it is independent of ω_n.

(iii) M_p is decreasing with increasing in ζ.

(iv) Smaller M_p corresponds to a system with better stability.

(v) Bandwidth is a function of both ζ and ω_n.

(vi) Bandwidth decreases with increase in ζ.

(vii) Smaller bandwidth correspond to an unstable system.

(viii) Smaller bandwidth gives better filtering of noise.

PART –C (10 MARKS)

1. **Derive Resonant peak and Resonant frequency.**

RESONANT PEAK (M)

Consider the closed loop transfer function of second order system,

$$\frac{C(s)}{R(s)} = M(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

The sinusoidal transfer function M(jω) is obtained by letting s = jω.

$$\therefore M(j\omega) = \frac{\omega_n^2}{(j\omega)^2 + 2\zeta\omega_n(j\omega) + \omega_n^2}$$

$$= \frac{\omega_n^2}{-\omega^2 + j2\zeta\omega_n\omega + \omega_n^2} = \frac{\omega_n^2}{\omega_n^2\left(-\frac{\omega^2}{\omega_n^2} + j2\zeta\frac{\omega}{\omega_n} + 1\right)} = \frac{1}{1 - \left(\frac{\omega}{\omega_n}\right)^2 + j2\zeta\frac{\omega}{\omega_n}}$$

Let, Normalized frequency, $u = \left(\frac{\omega}{\omega_n}\right)$

$$\therefore M(j\omega) = \frac{1}{(1-u^2) + j2\zeta u}$$

Let, M = Magnitude of closed loop transfer function

α = Phase of closed loop transfer function.

$$M = |M(j\omega)| = \left[\frac{1}{(1-u^2)^2 + (2\zeta u)^2}\right]^{\frac{1}{2}} = \left[(1-u^2)^2 + 4\zeta^2 u^2\right]^{-\frac{1}{2}}$$

$$\alpha = \angle M(j\omega) = -\tan^{-1}\frac{2\zeta u}{1-u^2}$$

The resonant peak is the maximum value of M. The condition for maximum value of M can be obtained by differentiating the equation of M with respect to u and letting dM/du = 0 when $u = u_r$,

where, $u_r = \frac{\omega_r}{\omega}$ = Normalized resonant frequency.

On differentiating equation with respect to u we get,

$$\frac{dM}{du} = \frac{d}{du}\left[(1-u^2)^2 + 4\zeta^2u^2\right]^{-\frac{1}{2}} = -\frac{1}{2}\left[(1-u^2)^2 + 4\zeta^2u^2\right]^{-\frac{3}{2}}\left[2(1-u^2)(-2u) + 8\zeta^2u\right]$$

$$= \frac{-\left[-4u(1-u^2) + 8\zeta^2u\right]}{2\left[(1-u^2)^2 + 4\zeta^2u^2\right]^{\frac{3}{2}}} = \frac{4u(1-u)^2 - 8\zeta^2u}{2\left[(1-u^2)^2 + 4\zeta^2u^2\right]^{\frac{3}{2}}}$$

Replace u by u_r in equation and equate to zero.

$$\frac{4u_r(1-u_r^2) - 8\zeta^2u_r}{2\left[(1-u_r^2)^2 + 4\zeta^2u_r^2\right]^{\frac{3}{2}}} = 0$$

The equation will be zero if numerator is zero. Hence, on equating numerator to zero we get,

$$4u_r(1-u_r^2) - 8\zeta^2 u_r = 0 \quad \Rightarrow \quad 4u_r - 4u_r^3 - 8\zeta^2 u_r = 0$$

$$\therefore\ 4u_r^3 = 4u_r - 8\zeta^2 u_r \quad \Rightarrow \quad u_r^2 = 1 - 2\zeta^2 \quad \Rightarrow \quad u_r = \sqrt{1-2\zeta^2}$$

Therefore, the resonant peak occurs when $u_r = \sqrt{1-2\zeta^2}$

Put this condition in the equation for M and solve for M_r.

$$\therefore M_r = \left.\frac{1}{\left[(1-u^2)^2 + 4\zeta^2 u^2\right]^{\frac{1}{2}}}\right|_{u=u_r} = \frac{1}{\left[(1-u_r^2)^2 + 4\zeta^2 u_r^2\right]^{\frac{1}{2}}} = \frac{1}{\left[(1-(1-2\zeta^2))^2 + 4\zeta^2(1-2\zeta^2)\right]^{\frac{1}{2}}}$$

$$= \frac{1}{\left[4\zeta^4 + 4\zeta^2 - 8\zeta^4\right]^{\frac{1}{2}}} = \frac{1}{\left[4\zeta^2 - 4\zeta^4\right]^{\frac{1}{2}}} = \frac{1}{\left[4\zeta^2(1-\zeta^2)\right]^{\frac{1}{2}}} = \frac{1}{2\zeta\sqrt{1-\zeta^2}}.$$

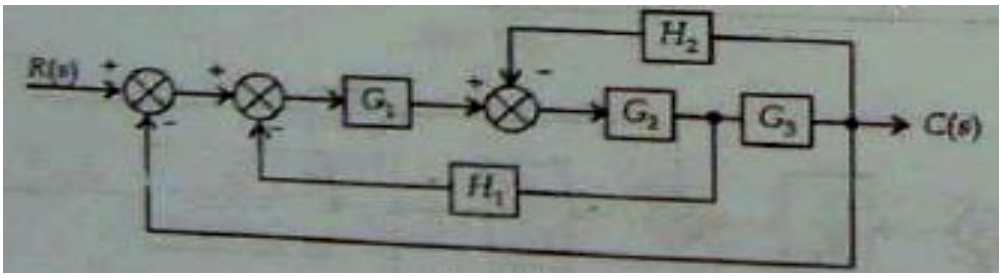
$$\therefore \boxed{\text{Resonant peak, } M_r = \frac{1}{2\zeta\sqrt{1-\zeta^2}}}$$

RESONANT FREQUENCY (ω_r)

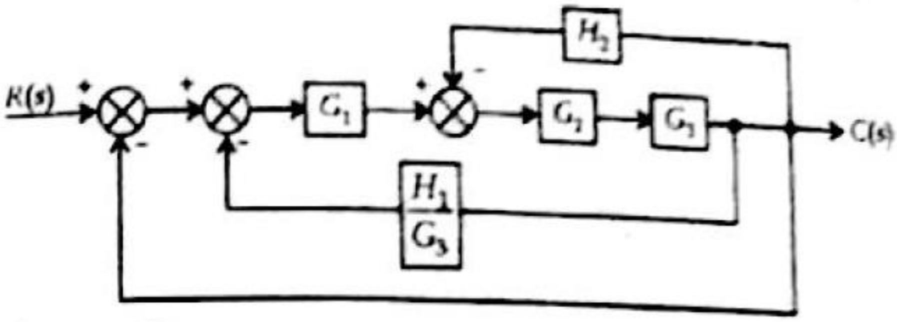
Normalized resonant frequency, $u_r = \dfrac{\omega_r}{\omega_n} = \sqrt{1-2\zeta^2}$

The resonant frequency, $\omega_r = \omega_n\sqrt{1-2\zeta^2}$

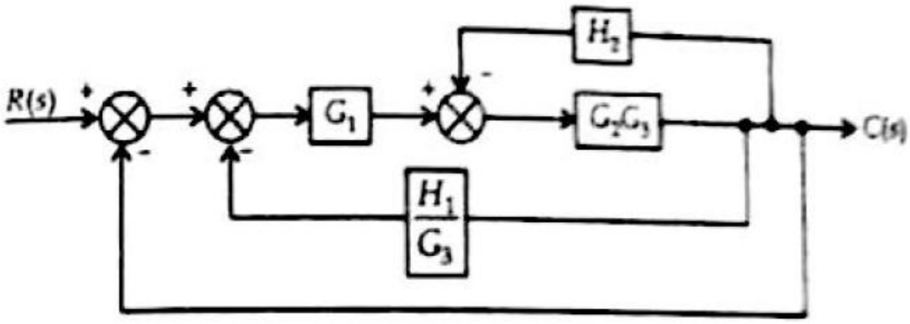
2. **Determine the Transfer Function for the given system.**



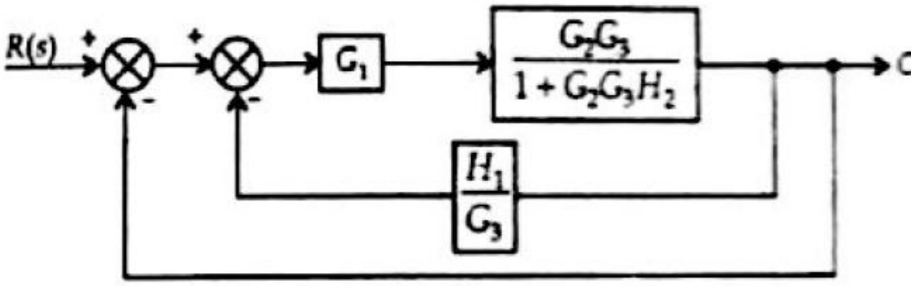
Solution : Step 1 : Shift the takeoff point beyond the block G_3



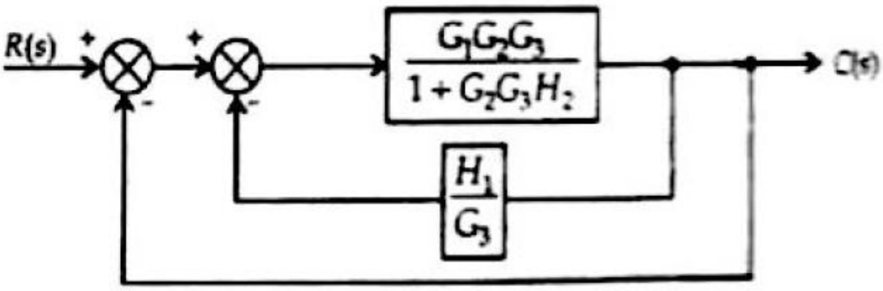
Step 2 :G_2 & G_3 are in cascade



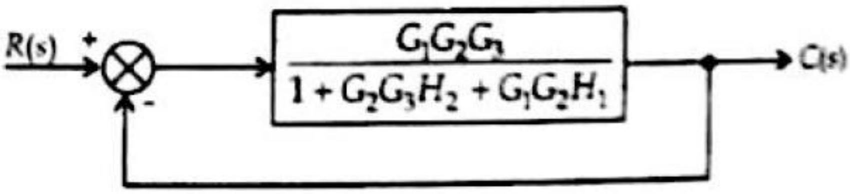
Step 3 :



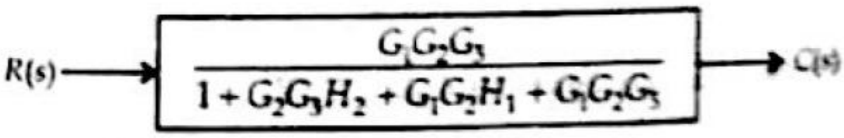
Step 4 :



Step 5 :



Step 6 :



$$\frac{C(s)}{R(s)} = \frac{G_1G_2G_3}{1 + G_2G_3H_2 + G_1G_2H_1 + G_1G_2G_3}$$ **Ans.**

3. $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$

$$: \quad \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

$$y = [0 \quad 1] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$: \text{Transfer function} = C\,[SI - A]^{-1}\,B$$

$$= [0 \quad 1] \begin{bmatrix} s+2 & 1 \\ 1 & s+2 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

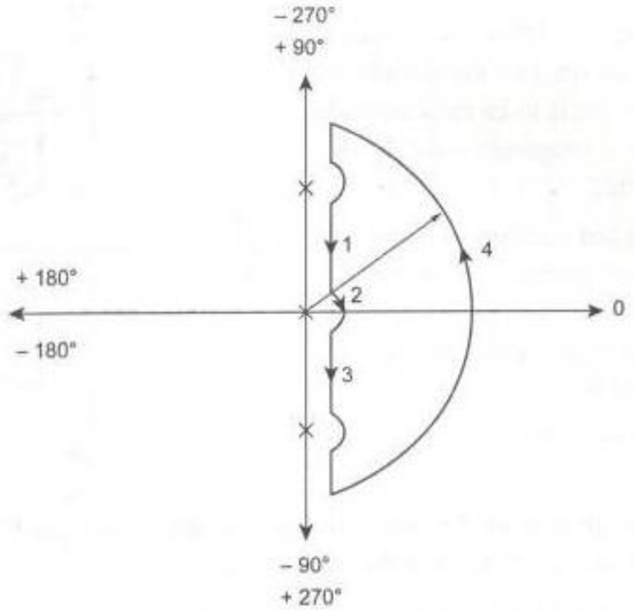
$$= [0 \quad 1] \frac{1}{(s+2)^2 - 1} \begin{bmatrix} s+2 & 1 \\ 1 & s+2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= [0 \quad 1] \frac{1}{s^2 + 4s + 3} \begin{bmatrix} s+2 \\ 1 \end{bmatrix}$$

$$\text{Transfer function} = [0 \quad 1] \begin{bmatrix} \dfrac{s+2}{s^2+4s+3} \\ \dfrac{1}{s^2+4s+3} \end{bmatrix}$$

$$\text{Transfer function} = \frac{1}{s^2 + 4s + 3}$$

4. Explain the procedure to draw Nyquist plot for stability analysis.



Step 4 : With the help of Nyquist path, draw the Nyquist path.

Step 5 : Select $(-1 + j0)$ point as reference point to do the closed loop system stability analysis.

Let p_{-1} = Number of closed loop poles that are on RHS of s-plane.

$p_{-1} = p_0$. It can be determined from given problem.

Let N_{-1} = Number of encirclements around $-1 + j0$ point.

Then $z_{-1} = N_{-1} + p_{-1}$

z_{-1} = The number of closed loop zeros that are on RHS of s-plane.

If $z_{-1} = 0$, the given system is stable.

$z_{-1} \neq 0$, the given system is unstable.

z_{-1} is the closed loop zeros that determine stability.

Nyquist criterion : For a closed loop system to be stable, G(S) H(S) plot must encircle $-1 + j0$ point as many number of times as the number of poles that are RHS of s-plane and encircles if any, must be made in clockwise direction.

5. **Consider the system described by the state model $\dot{X}$= AX and Y = CX where ,**
   A = $\begin{bmatrix} -1 & 1 \\ 1 & -2 \end{bmatrix}$, C = $\begin{bmatrix} 1 & 0 \end{bmatrix}$ Design a full order state observer the desired Eigen values for the observer matrix are μ_1 = -5 , μ_2 = -5

SOLUTION

Check for observability

Given that, $A = \begin{bmatrix} -1 & 1 \\ 1 & -2 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 0 \end{bmatrix}$

$$C^T = \begin{bmatrix} 1 & 0 \end{bmatrix}^T = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$A^T = \begin{bmatrix} -1 & 1 \\ 1 & -2 \end{bmatrix}^T = \begin{bmatrix} -1 & 1 \\ 1 & -2 \end{bmatrix} \quad ; \quad A^T C^T = \begin{bmatrix} -1 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\left.\begin{matrix}\text{Composite matrix} \\ \text{for observability}\end{matrix}\right\} Q_o = \begin{bmatrix} C^T & A^T C^T \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

$$\text{Determinant of } Q_o = \Delta_{Qo} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = 1$$

Since, $\Delta_{Qo} \neq 0$, the system is observable.

To determine the characteristic polynomial of original system

The characteristic polynomial of the system is given by $|\lambda I - A| = 0$.

$$[\lambda I - A] = \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} -1 & 1 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} \lambda + 1 & -1 \\ -1 & \lambda + 2 \end{bmatrix}$$

$$|\lambda I - A| = (\lambda + 1)(\lambda + 2) - 1 = \lambda^2 + 3\lambda + 2 - 1 = \lambda^2 + 3\lambda + 1$$

Hence the characteristic polynomial of original system is

$$\lambda^3 + 3\lambda + 1 = 0$$

To determine the desired characteristic polunomial

The deisred eigenvalues are $\mu_1 = -5$ and $\mu_2 = -5$

$$\therefore (\lambda - \mu_1)(\lambda - \mu_2) = (\lambda + 5)(\lambda + 5) = \lambda^2 + 10\lambda + 25.$$

The desired characteristic polynomial is

$$= \lambda^2 + 10\lambda + 25 = 0$$

$$|\lambda I - (A-GC)| = 0$$

Here, $G = \begin{bmatrix} g_1 \\ g_2 \end{bmatrix}$

$$[\lambda I - (A - GC)] = [\lambda I - A + GC]$$

$$= \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} -1 & 1 \\ 1 & -2 \end{bmatrix} + \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} -1 & 1 \\ 1 & -2 \end{bmatrix} + \begin{bmatrix} g_1 & 0 \\ g_2 & 0 \end{bmatrix} = \begin{bmatrix} \lambda + 1 + g_1 & -1 \\ -1 + g_2 & \lambda + 2 \end{bmatrix}$$

$$|\lambda I - A + GC| = \begin{vmatrix} \lambda + 1 + g_1 & -1 \\ -1 + g_2 & \lambda + 2 \end{vmatrix} = (\lambda + 1 + g_1)(\lambda + 2) - (-1 + g_2)(-1)$$

$$= \lambda^2 + 2\lambda + \lambda + 2 + g_1\lambda + 2g_1 - 1 + g_2$$

$$= \lambda^2 + (3 + g_1)\lambda + (2g_1 + g_2 + 1)$$

The characteristic polynomial of the systemn with state observer is

$$\lambda^2 + (3 + g_1)\lambda + (2g_1 + g_2 + 1) = 0$$

From equ(5.11.3) we get the desired characteristic polynomial of the system as

$$\lambda^2 + 10\lambda + 25 = 0$$

On comparing the coefficients of λ in equations (5.11.4) and (5.11.5) we get,

$$3 + g_1 = 0$$

$$\therefore g_1 = 10 - 3 = 7$$

On comparing the coefficients of λ^0 in equations (5.11.4) and (5.11.5) we get,

$$2g_1 + g_2 + 1 = 25$$

$$\therefore g_2 = 25 - 1 - 2g_1 = 24 - 2(7) = 10$$

$\therefore$ The observer gain matrix, $G = \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} = \begin{bmatrix} 7 \\ 10 \end{bmatrix}$

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PART –A (2 MARKS)

1. **What is state and state variable?**
The state is the condition of a system at any instant ,t. A set of variable which describes the state of the system at any time instant are called state variables.

2. **What is State transition matrix?**
The matrix exponential e^{At} is called state transition matrix.

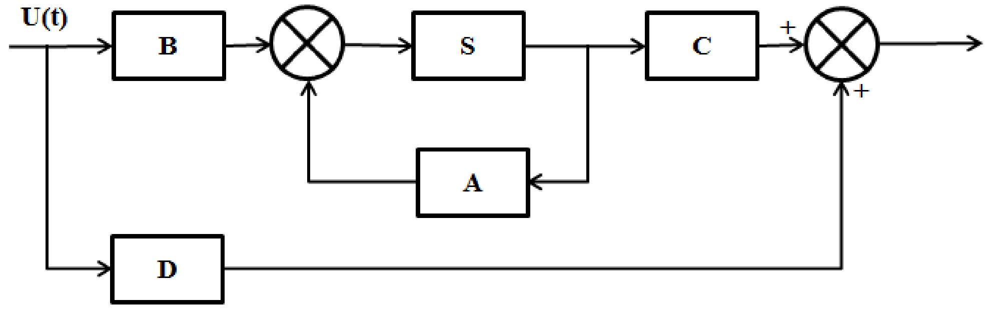
3. **Write the properties of state transition matrix.**
The following are the properties of state transition matrix
   1. Φ (0) = $e^{A(0)}$ = I Unit Matrix.
   2. Φ (t) = e^{At} = $(e^{-At})^{-1}$ = $[\Phi(-t)]^{-1}$.
   3. Φ (t1+t2) = $e^{A(t1+t2)}$ = Φ(t1) Φ(t2) = Φ(t2) Φ(t1).

4. **What is resolvant matrix ?**
The laplace transform of State transition matrix is called resolvant matrix.

5. **What is controllability and observability?**
A system is said to be completely state controllable if it is possible to transfer the system state from any initial state X(t0) at any other desired state X(t), in specified finite time by a control vector U(t).
A system is said to be completely observable if every state X(t) can be completely identified by measurements of the output Y(t) over a finite time interval.

6. **What is free response and forced response of a system ?**
Free response : It is the response of the system for zero input .
Forced response : Response of the system with excitate is called forced response.

7. **Draw the block diagram representation of state model .**
A block diagram of a system is a pictorial representation of the functions performed by each component of the system and shows the flow of signals. The basic elements of block diagram are block, branch point and summing point.



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8. Name the test signals used in control system
The commonly used test input signals in control system are impulse, step, ramp, acceleration and sinusoidal signals.

9. **How the system is classified depending on the value of damping?**
Depending on the value of damping, the system can be classified into the following four cases

Case 1 : Undamped system, $\tau = 0$
Case 2 : Underdamped system, $0 < \tau < 1$
Case 3 : Critically damped system, $\tau = 1$
Case 4 : Over damped system, $\tau > 1$.

10. What are the three constants associated with a steady state error?
The K_p, K_v and K_a are called static error constants. These constants are associated with steady state error in a particular type of system and for a standard input.

- Positional error constant
- Velocity error constant
- Acceleration error constant.

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PART –B (6 MARKS)

1. $A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$ **compute state transition matrix.**

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$$

$$e^{At}c = \phi(t) = \mathbf{L}^{-1}\left[(sI - A)^{-1}\right]$$

$$sI - A = s\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} = \begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix}$$

Let, $\Delta = |sI - A| =$ determinant of $(sI - A)$

$$\therefore \Delta = |sI - A| = \begin{vmatrix} s & -1 \\ 2 & s+3 \end{vmatrix} = s(s+3) + 2 = s^2 + 3s + 2 = (s+2)(s+1)$$

$$\phi(s) = [sI - A]^{-1} = \frac{[\text{Cofactor of } (sI - A)]^T}{\text{determinant of } (sI - A)} = \frac{[\text{Cofactor of } (sI - A)]^T}{\Delta}$$

$$\therefore \phi(s) = \frac{1}{(s+1)(s+2)}\begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix}$$

$$\phi(s) = \begin{bmatrix} \dfrac{s+3}{(s+1)(s+2)} & \dfrac{1}{(s+1)(s+2)} \\ \dfrac{-2}{(s+1)(s+2)} & \dfrac{s}{(s+1)(s+2)} \end{bmatrix}$$

By partial fraction expansion, $\phi(s)$ can be written as,

$$\phi(s) = \begin{bmatrix} \frac{A_1}{s+1} + \frac{B_1}{s+2} & \frac{A_2}{s+1} + \frac{B_2}{s+2} \\ \frac{A_3}{s+1} + \frac{B_3}{s+2} & \frac{A_4}{s+1} + \frac{B_4}{s+2} \end{bmatrix}$$

$$\frac{s+3}{(s+1)(s+2)} = \frac{A_1}{s+1} + \frac{B_1}{s+2}$$

$$A_1 = \left.\frac{s+3}{s+2}\right|_{s=-1} = 2$$

$$B_1 = \left.\frac{s+3}{s+1}\right|_{s=-2} = -1$$

$$\frac{1}{(s+1)(s+2)} = \frac{A_2}{s+1} + \frac{B_2}{s+2}$$

$$A_2 = \left.\frac{1}{s+2}\right|_{s=-1} = 1$$

$$B_2 = \left.\frac{1}{s+1}\right|_{s=-2} = -1$$

$$\frac{-2}{(s+1)(s+2)} = \frac{A_3}{s+1} + \frac{B_3}{s+2}$$

$$A_3 = \left.\frac{-2}{s+2}\right|_{s=-1} = -2$$

$$B_3 = \left.\frac{-2}{s+1}\right|_{s=-2} = 2$$

$$\frac{s}{(s+1)(s+2)} = \frac{A_4}{s+1} + \frac{B_4}{s+2}$$

$$A_4 = \left.\frac{s}{s+2}\right|_{s=-1} = -1$$

$$B_4 = \left.\frac{s}{s+1}\right|_{s=-2} = 2$$

$$\therefore \phi(s) = \begin{bmatrix} \frac{2}{s+1} - \frac{1}{s+2} & \frac{1}{s+1} - \frac{1}{s+2} \\ \frac{-2}{s+1} + \frac{2}{s+2} & \frac{-1}{s+1} + \frac{2}{s+2} \end{bmatrix}$$

On taking inverse Laplace transform of $\phi(s)$ we get $\phi(t)$, where $\phi(t) = e^{At}$

$$\therefore e^{At} = \phi(t) = \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix}$$

2. **Discuss the properties of state transition matrix.**

PROPERTIES OF STATE TRANSITION MATRIX

1. $\phi(0) = e^{A\times 0} = I$ (unit matrix)
2. $\phi(t) = e^{At} = \left(e^{-At}\right)^{-1} = [\phi(-t)]^{-1}$ or $\phi^{-1}(t) = \phi(-t)$
3. $\phi(t_1 + t_2) = e^{A(t_1+t_2)} = \left(e^{At_1}\right).\left(e^{At_2}\right) = \phi(t_1)\,\phi(t_2) = \phi(t_2)\,.\,\phi(t_1)$

PROPERTIES OF STATE TRANSITION MATRIX OF DISCRETE TIME SYSTEM

1. $\phi(0) = I$
2. $\phi^{-1}(k) = \phi(-k)$
3. $\phi(k, k_0) = \phi(k- k_0) = A^{(k-k_0)}$; where $k > k_0$

3. **Explain in detail about the design of state observer for continuous time systems**

4. **What is state & state variable? What is the need of state space analysis and state its advantages?**

The state is the condition of a system at any time instant, t. A set of variable which describes the state of the system at any time instant are called state variables.

The state space analysis is a modern approach and also easier for analysis using digital computers. The conventional (or old) methods of analysis employs the transfer function of the system.

The state space analysis is applicable to any type of systems. They can be used for modelling and analysis of linear & non-linear systems, time invariant & time variant systems and multiple input & multiple output systems.

The state space analysis can be performed with initial conditions.

The variables used to represent the system can be any variables in the system.

Using this analysis the internal states of the system at any time instant can be predicted.

5. **Derive the solution of homogenous state equations for free response.**

PART –C (10 MARKS)

1. **Determine the solution of state equation**

Here $A=\begin{bmatrix}1&0\\1&1\end{bmatrix}$; $sI-A=s\begin{bmatrix}1&0\\0&1\end{bmatrix}-\begin{bmatrix}1&0\\1&1\end{bmatrix}=\begin{bmatrix}s&0\\0&s\end{bmatrix}-\begin{bmatrix}1&0\\1&1\end{bmatrix}=\begin{bmatrix}s-1&0\\-1&s-1\end{bmatrix}$

$$|sI-A|=\begin{vmatrix}s-1&0\\-1&s-1\end{vmatrix}=(s-1)^2-0=(s-1)^2$$

$$(sI-A)^{-1}=\frac{1}{(s-1)^2}\begin{bmatrix}s-1&0\\1&s-1\end{bmatrix}=\begin{bmatrix}\frac{1}{s-1}&0\\\frac{1}{(s-1)^2}&\frac{1}{s-1}\end{bmatrix}$$

$$e^{At}=\phi(t)=\mathcal{L}^{-1}[(\phi(s)]=\mathcal{L}^{-1}\left[(sI-A)^{-1}\right]=\begin{bmatrix}e^t&0\\te^t&e^t\end{bmatrix}$$

The solution of the state equation is, $X(t)=e^{At}X_0=\begin{bmatrix}e^t&0\\te^t&e^t\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}e^t\\te^t\end{bmatrix}$

2. $\begin{bmatrix}\dot{x}_1\\\dot{x}_2\\\dot{x}_3\end{bmatrix}=\begin{bmatrix}0&0&1\\-2&-3&0\\0&2&-3\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}+\begin{bmatrix}0\\2\\0\end{bmatrix}[u]$

Y = $\begin{bmatrix}1&0&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$ **Check for controllability for kalman's test.**

From the given state model we get,

$$A = \begin{bmatrix} 0 & 0 & 1 \\ -2 & -3 & 0 \\ 0 & 2 & -3 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$$

$$A^2 = A.A = \begin{bmatrix} 0 & 0 & 1 \\ -2 & -3 & 0 \\ 0 & 2 & -3 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ -2 & -3 & 0 \\ 0 & 2 & -3 \end{bmatrix} = \begin{bmatrix} 0 & 2 & -3 \\ 6 & 9 & -2 \\ -4 & -12 & 9 \end{bmatrix}$$

$$A.B = \begin{bmatrix} 0 & 0 & 1 \\ -2 & -3 & 0 \\ 0 & 2 & -3 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -6 \\ 4 \end{bmatrix}$$

$$A^2.B = \begin{bmatrix} 0 & 2 & -3 \\ 6 & 9 & -2 \\ -4 & -12 & 9 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 18 \\ -24 \end{bmatrix}$$

The composite matrix for controllability, $Q_c = [B \quad AB \quad A^2B] = \begin{bmatrix} 0 & 0 & 4 \\ 2 & -6 & 18 \\ 0 & 4 & -24 \end{bmatrix}$

Determinant of $Q_c = \Delta_{QC} = \begin{vmatrix} 0 & 0 & 4 \\ 2 & -6 & 18 \\ 0 & 4 & -24 \end{vmatrix} = 4 \times 8 = 32.$

Since, $\Delta_{QC} \neq 0$, the rank of $Q_c = 3$. Hence the system is completely state controllable.

3. $$\begin{bmatrix}\dot{x_1}\\ \dot{x_2}\\ \dot{x_3}\end{bmatrix} = \begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 0 & -2 & -3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} + \begin{bmatrix}0\\ 0\\ 1\end{bmatrix}[u]$$

$$Y = [3 \quad 4 \quad 1]\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$$

Check for observability for kalman's test

Solution:

$$[X^{1}] = [A][X] + [B]u$$

$$A = \begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 0 & -2 & -3\end{bmatrix},\ B = \begin{bmatrix}0\\ 0\\ 1\end{bmatrix},\ C = [3 \quad 4 \quad 1]$$

Kalman's test for controllability:

$$Q_0 = [C^T \quad A^T C^T \quad (A^T)^2 C^T]$$

Note A^T = Transpose of matrix A.

A^T created by one of the following equivalent action.

Write the columns of A as the rows of A^T

[OR]

Write the rows of A as the columns of A^T

- Similar procedure is followed for C^T.

$$A^T C^T = \begin{bmatrix}0 & 0 & 0\\ 1 & 0 & -2\\ 0 & 1 & -3\end{bmatrix}\begin{bmatrix}3\\ 4\\ 1\end{bmatrix} = \begin{bmatrix}0\\ 1\\ 1\end{bmatrix}$$

$$(A^T)^2 C^T = A^T (A^T C^T) = \begin{bmatrix}0 & 0 & 0\\ 1 & 0 & -2\\ 0 & 1 & -3\end{bmatrix}\begin{bmatrix}0\\ 1\\ 1\end{bmatrix} = \begin{bmatrix}0\\ -2\\ -2\end{bmatrix}$$

$$Q_0 = [C^T \quad A^T C^T \quad (A^T)^2 C^T]$$

$$Q_0 = \begin{bmatrix}3 & 0 & 0\\ 4 & 1 & -2\\ 1 & 1 & -2\end{bmatrix}$$

$$|Q_0| = 0.$$

The determinant of Q_0 is zero. So the Q_0 is a singular matrix. Since the matrix Q_0 singular matrix, the given system is not completely state observable.

BEE003 Advanced Control System
Unit -II

4. **Determine the state model of the system determined by the differential equation.**

$x_1 = y, x_2 = \dot{y}$ and $x_3 = \ddot{y}$ then

$\dot{x}_1 = x_2$

$\dot{x}_2 = x_3$

$\dot{x}_3 = 8\,u(t) - 10\,x_1 - 11x_2 - 6x_3$

The last equation is obtained from the given equation.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -10 & -11 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 8 \end{bmatrix} u(t)$$

Compare equation (8.11) with equation (8.6) we get

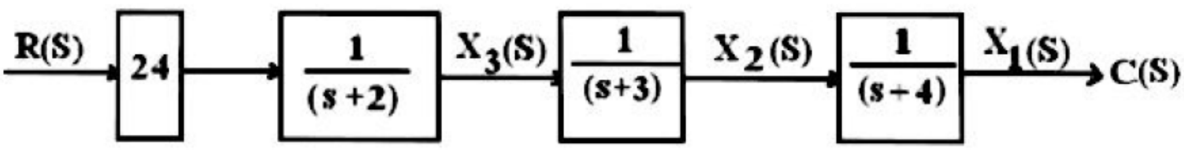
$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -10 & -11 & -6 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 8 \end{bmatrix}, \quad x(t) = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

5. **Obtain the state model and output for the transfer function by cascading.**

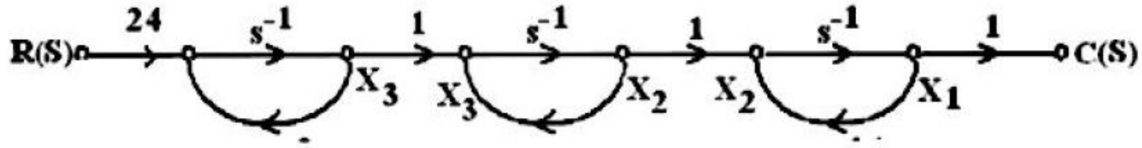
$$\frac{C(S)}{R(S)} = \frac{24}{s^3 + 9s^2 + 26s + 24}$$

$$\frac{C(S)}{R(S)} = \frac{24}{s^3 + 9s^2 + 26s + 24}$$

$$= \frac{24}{(s+2)(s+3)(s+4)}$$



$$\frac{X_1}{X_2} = \frac{1}{s+4} = \frac{s^{-1}}{1+4s^{-1}} = \frac{s^{-1}}{1-(-4s^{-1})}$$



$$\dot{X}_1 = X_2 - 4\ x\ 1$$

$$\dot{X}_2 = X_3 - 3\ x\ 1$$

$$\dot{X}_3 = -2X_3 + 24V$$

$$C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}$$

$$\begin{bmatrix} \dot{X}_1 \\ \dot{X}_2 \\ \dot{X}_3 \end{bmatrix} = \begin{bmatrix} -4 & 1 & 0 \\ 0 & -3 & 1 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 24 \end{bmatrix} R$$

BEE003 Advanced Control System
Unit -III
PART –A (2 MARKS)

1. **What is compensation?**
The compensation is the design procedure in which the system behaviour is altered to meet the desired specifications by introducing additional device called compensator.

2. **What is a lag compensator? Give an example?**
A compensator having the characteristics of lag network is called lag compensator. If a sinusoidal signal is applied to a lag compensator, then in steady state the output will have a phase lag and lead with respect to input.
An electrical lag compensator can be realised by an R-C network. The R-C network shown in the figure is an example of electrical lag compensator.

3. **What is lead compensator? Give an example.**
A compensator having the characteristic of a lead network is called a lead compensator. If a sinusoidal signal is applied to a lead compensator, then in steady state the output will have a phase lead with respect to input. An electrical lead compensator can be realised by a RC network. The R-C network shown in the figure is an example of electrical lead compensator.

4. **What is state estimator?**
A device that estimates i.e. observes the state variables is called state estimator.

5. **What are the uses of lag compensator?**
The lag compensator improves the steady state performance, reduces the bandwidth and increases the rise time. The increase in rise time results in slower transient response. If the pole introduced by the compensator is not cancelled by the zero in the system, then lag compensator increases the order of the system by one.

6. **What are the uses of lead compensator?**
The lead compensator increases the bandwidth and improves the speed of response. It also reduces the peak overshoot. If the pole introduced by the compensator is not cancelled by the zero in the system, then lead compensation increases the order of the system by one. When the given system is stable/ unstable and requires improvement in transient state response then lead compensation is employed.

7. **Why compensation is necessary in feedback control system?**
In feedback control systems compensation is required in the following Situations:
When the system is absolutely unstable, then compensation is required to stabilize the system and also to meet the desired performance.
When the system is stable, compensation is provided to obtain the desired performance.

8. **What are the factors to be considered or choosing compensation ?**
   - Nature of signals
   - Power levels at various points
   - Components available

9. **Discuss the effect of adding a pole to open loop transfer function of a system.**
   The addition of a pole to open loop transfer function of a system will reduce the steady state error. The closer the pole to origin lesser will be the steady-state error. Thus the steady- state performance of the system is improved. Also the addition of pole will increase the order of the system, which in turn makes the system less stable than the Original system.

10. **Discuss the effect of adding a zero to open loop transfer function of a system.**
    The addition of a zero to open loop transfer function of a system will improve the transient response. The addition of zero reduces the rise time. If the zero is introduced close to origin then the peak overshoot will be larger. If the zero is introduced far away from the origin in the left half of s-plane then the effect of zero on the transient response will be negligible.

BEE003 Advanced Control System

Unit -III

PART –B (6 MARKS)

1. **What is pole placement via state feedback? Give the necessary conditions for design using state feedback.**

The pole placement by state feedback is a control system design technique, in which the state variables are used for feedback to achieve the desired closed loop poles.

The state feedback design requires arbitrary pole placement to achieve the desired performance. The necessary and sufficient condition to be satisfied for arbitrary pole placement is that the system be completely state controllable.

2. **Compare led network and lag network.**

S.No.	Lead Network	Lag Network
i.	Phase lead network act as a differentiator.	Phase lag network act as a integrator
ii.	It acts as a high pass filter.	It acts as a low pass filter.
iii.	Phase lead network act as a pre emphasis circuit.	Phase lag network act as a dc circuit.
iv.	In Bode plot exist between corner frequencies.	In Bode plot to, exist between frequencies.
v.	In lead network design, zero exist near the origin, hence zero is the predominant factor.	In lag network design, pole exist net the origin, hence pole is the predominant factor.
vi.	In this network, settling time decreases.	In this network settling time rises.
vii.	In lead network, rise time decreases.	In this network, rise time increases
viii.	Bandwidth of the system increases through rise time t, decreases and thus the system becomes fast.	Bandwidth of the system dean through rise time 1, increases tie. speed is slow (or) this network dear-e rates the time response.
ix.	Signal to noise ratio is poor as it is a differentiator.	Signal to noise ratio is higher as at s integrator.
x.	Gain of the system increases.	Gain of the system decreases.
xi.	Lead network improves the transient response.	Lag network improves the steady response.

3. What is PI controller? What are its effects on system performance?

The PI-controller is a device that produces an output signal consisting of two terms-one proportional to input signal and the other proportional to the integral of input signal.

The introduction of PI-controller in the system reduces the steady state error and increases the order and type number of the system by one.

Transfer function of PI - controller $\left.\right\} G_c(s) = K_p + \frac{K_i}{s} = \frac{K_p\,(s + K_i / K_p)}{s}$

4. Discuss briefly about lag-lead compensation.

The lag-lead compensation is a design procedure in which a lag-lead compensator is introduced in the system so as to meet the desired specifications.

A compensator having the characteristics of lag-lead network is called lag-lead compensator. If a sinusoidal signal is applied to a lag-lead compensator then the output will have both phase lag and lead with respect to input, but in different frequency regions.

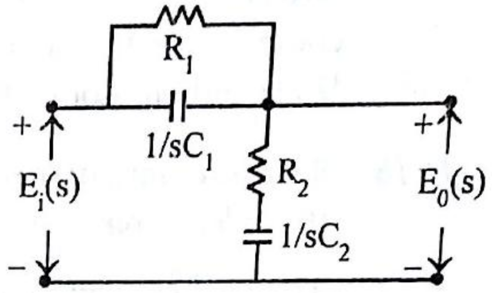


Fig Q1.22 : Electrical lag-lead compensator

An electrical lag-lead compensator can be realised by a R-C network. The R-C network shown in fig Q1.22 is an example of electrical lag-lead compensator.

5. Why compensation is required for a system? When feedback compensation is required for a system?

In feedback control systems compensation is required in the following situations.

1. When the system is absolutely unstable, then compensation is required to stabilize the system and also to meet the desired performance.

2. When the system is stable, compensation is provided to obtain the desired performance.

Lag compensation is employed for a stable system for improvement in steady state performance.

Lead compensation is employed for stable/unstable system for improvement in transient-state performance.

Lag-lead compensation is employed for stable/unstable system for improvement in both steady-state and transient state performance.

PART –C (10 MARKS)

1. Explain the steps for the design of lag compensation using bode plot.

PROCEDURE FOR THE DESIGN OF LAG COMPENSATOR USING BODE PLOT

The following steps may be followed to design a lag compensator using bode plot and to be connected in series with transfer function of uncompensated system, G(s).

Step-1 : Choose the value of K in uncompensated system to meet the steady state error requirement.

Step-2 : Sketch the bode plot of uncompensated system. *[Refer Appendix-I for the procedure to sketch bode plot].*

Step-3 : Determine the phase margin of the uncompensated system from the bode plot. If the phase margin does not satisfy the requirement then lag compensation is required.

Step-4 : Choose a suitable value for the phase margin of the compensated system.

Let, γ_d = Desired phase margin as given in specifications.

and γ_n = Phase margin of compensated system.

Now, $\gamma_n = \gamma_d + \epsilon$

Where ϵ = Additional phase lag to compensate for shift in gain crossover frequency.

Choose an initial value of $\epsilon = 5°$.

Step-5 : Determine the new gain crossover frequency, ω_{gcn}. The new ω_{gcn} is the frequency corresponding to a phase margin of γ_n on the bode plot of uncompensated system.

Let, ϕ_{gcn} = Phase of G(jω) at new gain crossover frequency, ω_{gcn}

Now, $\gamma_n = 180 + \phi_{gcn}$ (or) $\phi_{gcn} = \gamma_n - 180°$

The new gain crossover frequency, ω_{gcn} is given by the frequency at which the phase of G(jω) is ϕ_{gcn}.

Step-6 : Determine the parameter, β of the compensator. The value of β is given by the magnitude of G(jω) at new gain crossover frequency, ω_{gcn}. Find the db gain at new gain crossover frequency, ω_{gcn}

$$\text{Now, } A_{gcn} = 20\log\beta \text{ (or) } \frac{A_{gcn}}{20} = \log\beta, \quad \therefore \beta = 10^{A_{gcn}/20}$$

Step-7 : Determine the transfer function of lag compensator.
Place the zero of the compensator arbitrarily at 1/10th of the new gain crossover frequency, ω_{gcn}.

$$\therefore \text{Zero of the lag compensator, } z_c = \frac{1}{T} = \frac{\omega_{gcn}}{10}$$

$$\text{Now, } T = \frac{10}{\omega_{gcn}}.$$

Pole of the lag compensator, $p_c = 1/\beta T$

$$\left.\begin{array}{l}\text{Transfer function}\\ \text{of lag compensator}\end{array}\right\} G_c(s) = \frac{s+\frac{1}{T}}{s+\frac{1}{\beta T}} = \beta\left(\frac{1+sT}{1+s\beta T}\right)$$

Step-8 : Determine the open loop transfer function of compensated system. The lag compensator is connected in series with the plant as shown in fig 1.7.

When the lag compensator is inserted in series with the plant, the open loop gain of the system is amplified by the factor β(∵ β>1). If the gain produced is not required then attenuator with gain 1/β can be introduced in series with the lag compensator to nullify the gain produced by lag compensator.

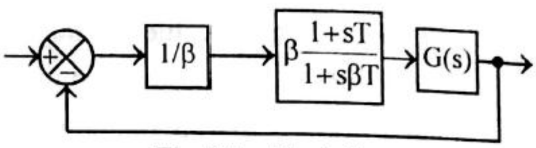


Fig 1.7 : Block diagram of lag compensated system

The open loop transfer function of the compensated system,

$$G_o(s) = \frac{1}{\beta}.G_c(s).G(s) = \frac{1}{\beta}.\beta\frac{(1+sT)}{(1+s\beta T)}.G(s) = \frac{(1+sT)}{(1+s\beta T)} \cdot G(s)$$

Step-9 : Determine the actual phase margin of compensated system. Calculate the actual phase angle of the compensated system using the compensated transfer function at new gain crossover frequency, ω_{gcn}.

Let, ϕ_{gco} = Phase of $G_o(j\omega)$ at $\omega = \omega_{gcn}$

Actual phase margin of the compensated system, $\gamma_o = 180° + \phi_{gco}$
If the actual phase margin satisfies the given specification then the design is accepted. Otherwise repeat the procedure from step 4 to 10 by taking ∈ as 5° more than previous design.

2. $\begin{bmatrix}\dot{x_1}\\ \dot{x_2}\\ \dot{x_3}\end{bmatrix}=\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 0 & 2 & -3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}+\begin{bmatrix}0\\ 0\\ 10\end{bmatrix}[u]$ **and composite matrix** $Q_c = \begin{bmatrix}0 & 0 & 10\\ 0 & 10 & -30\\ 10 & -30 & 70\end{bmatrix}$

Design a feedback controller with state feedback so that the closed loop poles are placed at -2, -1±j1

<u>Check for Controllability</u>

Given that, $A = \begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 0 & -2 & -3\end{bmatrix}$ and $B = \begin{bmatrix}0\\ 0\\ 10\end{bmatrix}$

$$A^2 = A.A = \begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 0 & -2 & -3\end{bmatrix}\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 0 & -2 & -3\end{bmatrix} = \begin{bmatrix}0 & 0 & 1\\ 0 & -2 & -3\\ 0 & 6 & 7\end{bmatrix}$$

$$AB = \begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 0 & -2 & -3\end{bmatrix}\begin{bmatrix}0\\ 0\\ 10\end{bmatrix} = \begin{bmatrix}0\\ 10\\ -30\end{bmatrix}$$

$$A^2B = \begin{bmatrix}0 & 0 & 1\\ 0 & -2 & -3\\ 0 & 6 & 7\end{bmatrix}\begin{bmatrix}0\\ 0\\ 10\end{bmatrix} = \begin{bmatrix}10\\ -30\\ 70\end{bmatrix}$$

Composite matrix for controllability, $Q_c = \begin{bmatrix}B & AB & A^2B\end{bmatrix} = \begin{bmatrix}0 & 0 & 10\\ 0 & 10 & -30\\ 10 & -30 & 70\end{bmatrix}$

Determinant of $Q_c = \Delta_{Qc} = \begin{vmatrix}0 & 0 & 10\\ 0 & 10 & -30\\ 10 & -30 & 70\end{vmatrix} = 10(-10x10) = -1000$

Since, $\Delta_{Qc} \neq 0$, the system is completely state controllable.

To find Q_C^{-1}

From equation (5.8.6) and (5.8.7) we get

$$Q_c = \begin{bmatrix} 0 & 0 & 10 \\ 0 & 10 & -30 \\ 10 & -30 & 70 \end{bmatrix} \text{ and } \Delta_{QC} = -1000.$$

$$Q_c^{-1} = \frac{[\text{cofactor of } Q_c]^T}{\text{Determinant of } Q_c} = \frac{1}{\Delta_{Qc}} \begin{bmatrix} 610 & -300 & -100 \\ 300 & -100 & 0 \\ -100 & 0 & 0 \end{bmatrix}^T$$

$$= \frac{1}{-1000} \begin{bmatrix} 610 & 300 & -100 \\ -300 & -100 & 0 \\ -100 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -0.61 & 0.3 & 0.1 \\ -0.3 & 0.1 & 0 \\ 0.1 & 0 & 0 \end{bmatrix}$$

To find desired characteristic polynomial

The desired closed loop poles are

$$\mu_1 = -2, \mu_2 = -1 + j1 \text{ and } \mu_3 = -1 - j1$$

Hence the desired characteristic polynomial is

$$(\lambda - \mu_1)(\lambda - \mu_2)(\lambda - \mu_3) = (\lambda + 2)(\lambda + 1 - j1)(\lambda + 1 + j1)$$

$$= (\lambda + 2)\left((\lambda + 1)^2 - (j1)^2\right)$$

$$= (\lambda + 2)\left(\lambda^2 + 2\lambda + 1 + 1\right)$$

$$= (\lambda + 2)\left(\lambda^2 + 2\lambda + 2\right)$$

$$= \lambda^3 + 2\lambda^2 + 2\lambda + 2\lambda^2 + 4\lambda + 4$$

$$= \lambda^3 + 4\lambda^2 + 6\lambda + 4$$

The desired characteristic polynomial is

$$\lambda^3 + 4\lambda^2 + 6\lambda + 4 = 0$$

To determine the state variable feedback matrix, K

Method - I

The characteristic polynomial of original system is given by $|\lambda I - A| = 0$

$$[\lambda I - A] = \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & -3 \end{bmatrix} = \begin{bmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ 0 & 2 & \lambda+3 \end{bmatrix}$$

$$|\lambda I - A| = \begin{vmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ 0 & 2 & \lambda+3 \end{vmatrix} = \lambda\,[\lambda\,(\lambda+3)+2] = \lambda^2(\lambda+3)+2\lambda = \lambda^3+3\lambda^2+2\lambda.$$

The characteristic polynomial of original system is,

$$\lambda^3 + 3\lambda^2 + 2\lambda = 0 \qquad .....(5.8.10)$$

From equ(5.8.9) we get the desired characteristic polynomial as

$$\lambda^3 + 4\lambda^2 + 6\lambda + 4 = 0 \qquad .....(5.8.11)$$

From equation (5.8.8) we get,

$$Q_C^{-1} = \begin{bmatrix} -0.61 & 0.3 & 0.1 \\ -0.3 & 0.1 & 0 \\ 0.1 & 0 & 0 \end{bmatrix}$$

$$P_1 = [0 \quad 0 \quad 1]\, Q_c^{-1} = [0 \; 0 \; 1] \begin{bmatrix} -0.61 & 0.3 & 0.1 \\ -0.3 & 0.1 & 0 \\ 0.1 & 0 & 0 \end{bmatrix} = [0.1 \quad 0 \quad 0]$$

$$P_1 A = [0.1 \; 0 \; 0] \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & -3 \end{bmatrix} = [0 \; 0.1 \; 0]$$

$$P_1 A^2 = [0.1 \; 0 \; 0] \begin{bmatrix} 0 & 0 & 1 \\ 0 & -2 & -3 \\ 0 & 6 & 7 \end{bmatrix} = [0 \; 0 \; 0.1]$$

$$\therefore P_c = \begin{bmatrix} P_1 \\ P_1A \\ P_1A^2 \end{bmatrix} = \begin{bmatrix} 0.1 & 0 & 0 \\ 0 & 0.1 & 0 \\ 0 & 0 & 0.1 \end{bmatrix}$$

The state feedback gain matrix, $K = [\alpha_3 - a_3 \quad \alpha_2 - a_2 \quad \alpha_1 - a_1]\, P_c$

From equation (5.8.11) we get, $\alpha_3 = 4$; $\alpha_2 = 6$; $\alpha_1 = 4$

From equation (5.8.10) we get, $a_3 = 0$; $a_2 = 2$; $a_1 = 3$

$$\therefore K = [4-0 \quad 6-2 \quad 4-3] P_c$$

$$= [4 \ 4 \ 1] \begin{bmatrix} 0.1 & 0 & 0 \\ 0 & 0.1 & 0 \\ 0 & 0 & 0.1 \end{bmatrix} = [0.4 \quad 0.4 \quad 0.1]$$

3. Explain the steps for the design of lead compensation using bode plot.

PROCEDURE FOR DESIGN OF LEAD COMPENSATOR USING BODE PLOT

The following steps may be followed to design a lead compensator using bode plot and to be connected in series with transfer function of uncompensated system, G(s).

Step-1 : The open loop gain K of the given system is determined to satisfy the requirement of the error constant.

Step-2 : The bode plot is drawn for the uncompensated system using the value of K, determined from the previous step. *[Refer Appendix-I for the procedure to sketch bode plot].*

Step-3 : The phase margin of the uncompensated system is determined from the bode plot.

Step-4 : Determine the amount of phase angle to be contributed by the lead network by using the formula given below,

$$\phi_m = \gamma_d - \gamma + \epsilon$$

where,

ϕ_m = Maximum phase lead angle of the lead compensator

γ_d = Desired phase margin

γ = Phase margin of the uncompensated system

ϵ = Additional phase lead to compensate for shift in gain crossover frequency

Choose an initial choice of ϵ as 5^0

(Note : If ϕ_m is more than 60° then realize the compensator as cascade of two lead compensator with each compensator contributing half of the required angle).

Step-5 : Determine the transfer function of lead compensator

Calculate α using the equation, $\alpha = \dfrac{1 - \sin\phi_m}{1 + \sin\phi_m}$

From the bode plot, determine the frequency at which the magnitude of $G(j\omega)$ is $-20 \log 1/\sqrt{\alpha}$ db. This frequency is ω_m.

Calculate T from the relation, $\omega_m = \dfrac{1}{T\sqrt{\alpha}}$ $\qquad \therefore T = \dfrac{1}{\omega_m\sqrt{\alpha}}$

Transfer function of lead compensator $\Big\}$ $G_c(s) = \dfrac{s + \frac{1}{T}}{s + \frac{1}{\alpha T}} = \dfrac{\alpha(1 + sT)}{(1 + \alpha sT)}$

Step-6 : Determine the open loop transfer function of compensated system.

The lag compensator is connected in series with G(s) as shown in fig 1.12. When the lead network is inserted in series with the plant, the open loop gain of the system is attenuated by the factor α (∵ α<1), so an amplifier with the gain of 1/α has to be introduced in series with the compensator to nullify the attenuation caused by the lead compensator.

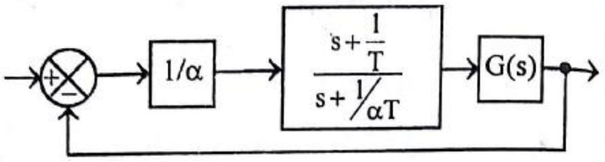


Fig 1.12 : Block diagram of lead compensated system

Open loop transfer function of the overall system $\Big\}$ $G_0(s) = \frac{1}{\alpha} \frac{s+\frac{1}{T}}{s+\frac{1}{\alpha T}}.G(s)$

$$= \frac{1}{\alpha} \cdot \frac{\alpha(1+sT)}{(1+s\alpha T)}.G(s) = \frac{(1+sT)}{(1+s\alpha T)}.G(s)$$

Step-7 : Verify the design.

Finally the Bode plot of the compensated system is drawn and verify whether it satisfies the given specifications. If the phase margin of the compensated system is less than the required phase margin then repeat step 4 to 10 by taking $\in$ as 5^0 more than the previous design.

4. A unity feedback system has an open loop transfer function $G(s) = \frac{K}{S(1+2S)}$. Determine the poles and zeros for lag compensator, so that phase margin is 40^0 and steady state error for ramp input is less than or equal to 0.2.

SOLUTION

Step-1: Calculation of gain, K.

Given that, $e_{ss} \leq 0.2$ for ramp input ; Let $e_{ss} = 0.2$

We know that, $e_{ss} = 1/K_v$ for ramp input

$\therefore$ Velocity error constant, $K_v = \frac{1}{e_{ss}} = \frac{1}{0.2} = 5$.

By definition of velocity error constant, $K_v = \underset{s\to 0}{Lt}\ s\,G(s)H(s)$.

Since the system is unity feedback system, H(s) = 1.

$$\therefore K_v = \underset{s\to 0}{Lt}\ s\,G(s) = \underset{s\to 0}{Lt}\ s\frac{K}{s(1+2s)} = K \qquad \therefore K = 5$$

Step-2: Bode plot of uncompensated system.

Given that, $G(s) = 5/s(1+2s)$

Let $s = j\omega$, $\therefore G(j\omega) = 5/j\omega(1+j2\omega)$.

Magnitude plot

The corner frequency is, $\omega_c = 1/2 = 0.5$ rad/sec

The various terms of $G(j\omega)$ are listed in table-1. Also the table shows the slope contributed by each term and the change in slope at the corner frequency.

TABLE-1

Term	Corner frequency rad/sec	Slope db/dec	Change in slope db/dec
$\frac{5}{j\omega}$	-	-20	-
$\frac{1}{1+j2\omega}$	$\omega_c = \frac{1}{2} = 0.5$	-20	$-20-20 = -40$

Choose a low frequency ω_l such that $\omega_l < \omega_c$ and choose a high frequency ω_h such that $\omega_h > \omega_c$.

Let $\omega_l = 0.1$ rad/sec and $\omega_h = 10$ rad/sec

Let $A = |G(j\omega)|$ in db

At $\omega = \omega_l$, $A = 20 \log\left|\frac{5}{j\omega}\right| = 20 \log \frac{5}{0.1} = 34$ db

At $\omega = \omega_c$, $A = 20 \log\left|\frac{5}{j\omega}\right| = 20 \log \frac{5}{0.5} = 20$ db

At $\omega = \omega_h$, $A = \left[\text{slope from } \omega_c \text{ to } \omega_h \times \log \frac{\omega_h}{\omega_c}\right] + A_{(\text{at } \omega = \omega_c)}$

$$= -40 \times \log \frac{10}{0.5} + 20 = -32 \text{ db}$$

Let the points a, b and c be the points corresponding to frequencies ω_l, ω_c and ω_h respectively on the magnitude plot. In a semilog graph sheet choose appropriate scales and fix the points a, b and c. Join the points by straight lines and mark the slope on the respective region. The magnitude plot is shown in fig 1.1.1.

Phase Plot

The phase angle of G(jω) as a function of ω is given by

$$\phi = \angle G(j\omega) = -90^\circ - \tan^{-1} 2\omega$$

The phase angle of G(jω) are calculated for various values of ω and listed in table-2.

TABLE-2

ω rad/sec	0.1	0.5	1.0	5	10
φ deg	−101	−135	−153	−174	−177

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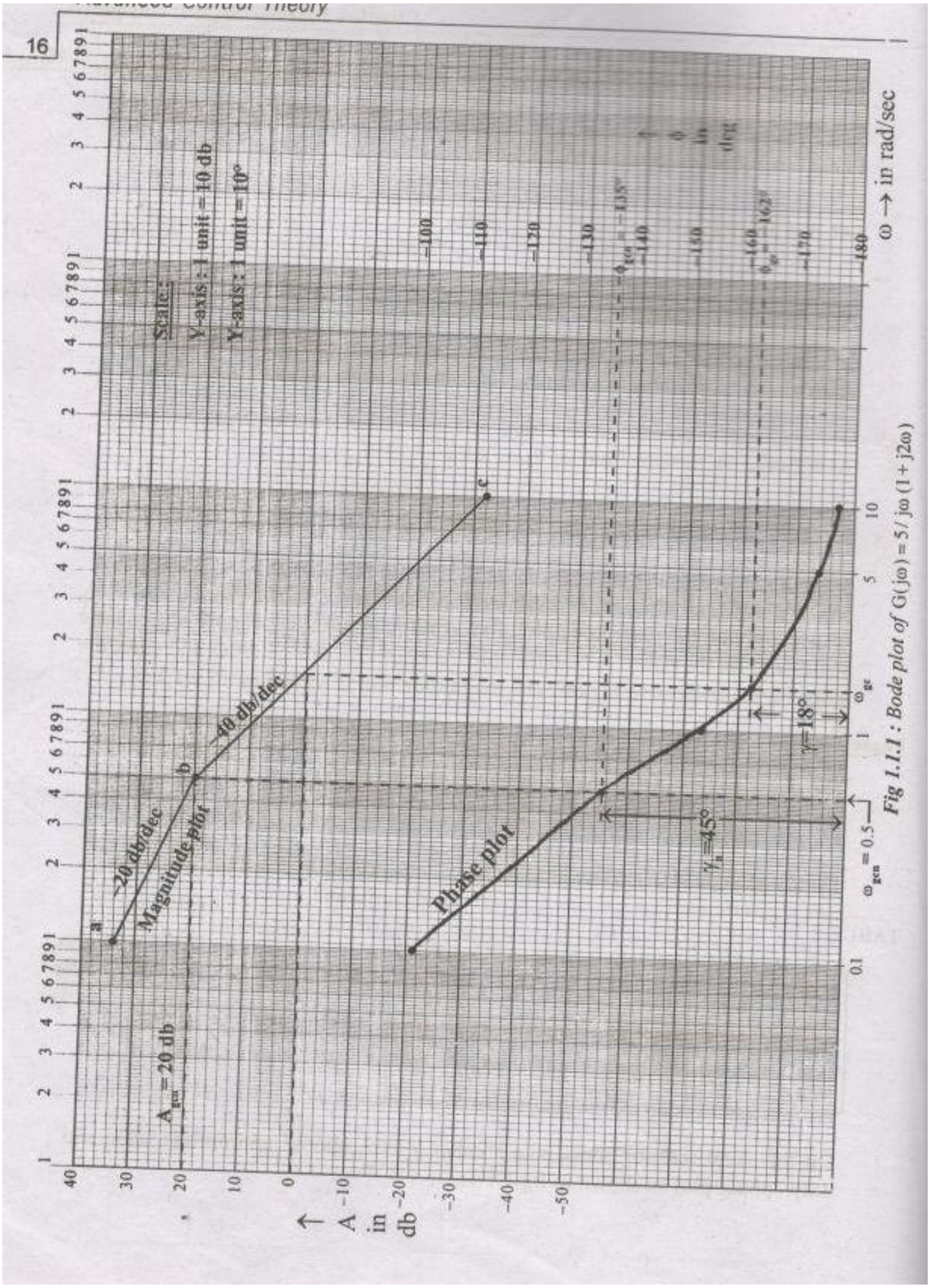


Fig 1.1.1 : Bode plot of $G(j\omega) = 5/ j\omega\,(1 + j2\omega)$

Step-3 : Determination of phase margin of uncompensated system.

Let, ϕ_{gc} = Phase of $G(j\omega)$ at gain crossover frequency (ω_{gc}).

and γ = Phase margin of uncompensated system.

From the bode plot of uncompensated system we get, $\phi_{gc} = -162°$.

Now, $\gamma = 180° + \phi_{gc} = 180° - 162° = 18°$

The system requires a phase margin of 40°, but the available phase margin is 18° and so lag compensation should be employed to improve the phase margin.

Step-4 : Choose a suitable value for the phase margin of compensated system.

The desired phase margin, $\gamma_d = 40°$.

$\therefore$ Phase margin of compensated system, $\gamma_n = \gamma_d + \in$

Let initial choice of $\in = 5°$

$$\therefore \gamma_n = \gamma_d + \in = 40° + 5° = 45°$$

Step 5 : Determine new gain crossover frequency.

Let ω_{gcn} = New gain crossover frequency and ϕ_{gcn} = Phase of $G(j\omega)$ at ω_{gcn}

Now, $\gamma_n = 180° + \phi_{gcn}$

$$\therefore \phi_{gcn} = \gamma_n - 180 = 45° - 180° = -135°$$

From the bode plot we found that, the frequency corresponding to a phase of −135° is 0.5 rad/sec.

$\therefore$ New gain crossover frequency, $\omega_{gcn} = 0.5$ rad/sec.

Step-6 : Determine the parameter, β

From the bode plot we found that, the db magnitude at ω_{gcn} is 20 db.

$\therefore |G(j\omega)|$ in db at $(\omega = \omega_{gcn}) = A_{gcn} = 20$ db

Also, $A_{gcn} = 20 \log \beta$; $\therefore \beta = 10^{A_{gcn}/20} = 10^{20/20} = 10.$

Step-7 : Determine the transfer function of lag compensator.

The zero of the compensator is placed at a frequency one-tenth of ω_{gcn}.

$$\therefore \text{Zero of the lag compensator, } z_c = \frac{1}{T} = \frac{\omega_{gcn}}{10}$$

$$\text{Now, } T = \frac{10}{\omega_{gcn}} = \frac{10}{0.5} = 20$$

$$\text{Pole of the lag compensator, } p_c = \frac{1}{\beta T} = \frac{1}{10 \times 20} = \frac{1}{200} = 0.005$$

Transfer function of lag compensator, $G_c(s) = \dfrac{s+\frac{1}{T}}{s+\frac{1}{\beta T}} = \beta\dfrac{1+sT}{1+s\beta T} = 10\dfrac{(1+20s)}{(1+200s)}$

Step-8 : Determine the open loop transfer function of compensated system.

The block diagram of the compensated system is shown in fig 1.1.2. The gain of the compensator is nullified by introducing an attenuator in series with the compensator, as shown in fig 1.1.2.

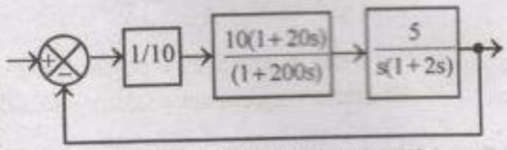


Fig 1.1.2 : Block diagram of lag compensated system

Open loop transfer function of compensated system $\Big\}\; G_o(s) = \dfrac{1}{10} \times \dfrac{10\,(1+20s)}{(1+200s)} \times \dfrac{5}{s(1+2s)}$

$$= \frac{5(1+20s)}{s(1+200s)\,(1+2s)}$$

Step-9 : Determine the actual phase margin of compensated system.

On substituting $s = j\omega$ in $G_o(s)$ we get, $G_o(j\omega) = \dfrac{5\,(1+j20\omega)}{j\omega\,(1+j200\omega)\,(1+j2\omega)}$

Let, ϕ_o = Phase of $G_o(j\omega)$

and ϕ_{gco} = Phase of $G_o(j\omega)$ at $\omega = \omega_{gcn}$

$\phi_o = \tan^{-1} 20\omega - 90° - \tan^{-1} 200\omega - \tan^{-1} 2\omega$

At $\omega = \omega_{gcn}$, $\phi_o = \phi_{gco} = \tan^{-1} 20\,\omega_{gcn} - 90° - \tan^{-1} 200\omega_{gcn} - \tan^{-1} 2\,\omega_{gcn}$

$\therefore \phi_{gco} = \tan^{-1}(20 \times 0.5) - 90° - \tan^{-1}(200 \times 0.5) - \tan^{-1}(2 \times 0.5) = -140°.$

Actual phase margin of compensated system, $\gamma_o = 180° + \phi_{gco}$

$$= 180° - 140° = 40°$$

CONCLUSION

The actual phase margin of the compensated system satisfies the requirement. Hence the design is acceptable.

RESULT

Transfer function of lag compensator, $G_c(s) = \dfrac{10(1+20s)}{(1+200s)} = \dfrac{(s+0.05)}{(s+0.005)}$

Open loop transfer function of compensated system, $G_o(s) = \dfrac{5\,(1+20s)}{s\,(1+200s)\,(1+2s)}$

5. Design a phase lead compensator for the unity feedback system $G(s) = \frac{K}{S(1+S)}$ to satisfy the following specifications.

SOLUTION

Step-1 : Determine K

Given that, steady state error, $e_{ss} \leq 1/15$ for unit ramp input

When the input is unit ramp, $e_{ss} = 1/K_v = 1/15$. $\therefore K_v = 15$

By definition of velocity error constant, K_v we get,

$$K_v = \underset{s \to 0}{Lt} \ s.G(s) H(s)$$

Here, $G(s) = \dfrac{K}{s(s+1)}$ and $H(s) = 1$, $\therefore K_v = \underset{s \to 0}{Lt} \ s.\dfrac{K}{s(s+1)} = K$

Step-2 : Draw bode plot

Given that, $G(s) = \dfrac{K}{s(s+1)} = \dfrac{15}{s(s+1)}$

Let $s = j\omega$, $\therefore G(j\omega) = \dfrac{15}{j\omega(1+j\omega)}$

Magnitude plot

The corner frequency is, $\omega_{c1} = 1$ rad/sec.

The various terms of $G(j\omega)$ are listed in table-1. Also the table shows the slope contributed by each term and the change in slope at the corner frequency.

TABLE-1

Term	Corner frequency rad/sec	Slope db/dec	Change in slope db/dec
$\dfrac{15}{j\omega}$	-	-20	-
$\dfrac{1}{1+j\omega}$	$\omega_{c1} = 1$	-20	$-20 + (-20) = -40$

Choose a low frequency ω_l such that $\omega_l < \omega_{c1}$ and choose a high frequency ω_h such, that $\omega_h > \omega_{c1}$.

Let $\omega_l = 0.1$ rad/sec and $\omega_h = 10$ rad/sec

Let $A = |G(j\omega)|$ in db

Let us calculate A at ω_l, ω_{c1} and ω_h.

At $\omega = \omega_l = 0.1$ rad / sec, $A = 20 \log\left|\frac{15}{j\omega}\right| = 20 \log \frac{15}{0.1} = 43.5 \text{ db} \approx 44 \text{ db}$

At $\omega = \omega_{c1} = 1$ rad / sec, $A = 20 \log\left|\frac{15}{j\omega}\right| = 20 \log \frac{15}{1} = 23.5 \text{ db} \approx 24 \text{ db}$

At $\omega = \omega_h = 10$ rad / sec, $A = \left[\text{slope from } \omega_{c1} \text{ to } \omega_h \times \log\frac{\omega_h}{\omega_{c1}}\right] + A_{(\text{at } \omega = \omega_{c1})}$

$$= -40 \times \log\frac{10}{1} + 24 = -16 \text{ db}$$

Let the points a, b and c be the points corresponding to frequencies ω_l, ω_{c1}, and ω_h respectively on the magnitude plot. In a semilog graph sheet choose appropriate scales and fix the points a, b and c. Join the points by straight lines and mark the slope on the respective region. The magnitude plot is shown in fig 1.5.2.

Phase Plot

The phase angle of G(jω) as a function of ω is given by

$$\phi = \angle G(j\omega) = -90° - \tan^{-1}\omega$$

The phase angle of G(jω) are calculated for various values of ω and listed in table-2.

TABLE-2

ω rad/sec	0.1	0.5	1	2	5	10
φ deg	−96	−117	−135	−153	−169	−174

On the same semilog sheet take another y-axis, choose appropriate scale and draw phase plot as shown in fig 1.5.2.

Step-3 : Determine the phase margin of uncompensated system.

Let, ϕ_{gc} = Phase of $G(j\omega)$ at gain crossover frequency (ω_{gc}).

and γ = Phase margin of uncompensated system.

From the bode plot of uncompensated system we get, $\phi_{gc} = -167°$.

Now, $\gamma = 180° + \phi_{gc} = 180° - 167° = 13°$

The system requires a phase margin of 45°, but the available phase margin is 13° and so lead compensation should be employed to improve the phase margin.

Step-4 : Find ϕ_m

The desired phase margin, $\gamma_d \geq 45°$

Let additional phase lead required, $\epsilon = 5°$

Maximum lead angle, $\phi_m = \gamma_d - \gamma + \epsilon = 45° - 13° + 5° = 37°$

Step-5 : Determine the transfer function of lead compensator.

$$\alpha = \frac{1-\sin\phi_m}{1+\sin\phi_m} = \frac{1-\sin 37°}{1+\sin 37°} = 0.2486 \approx 0.25$$

The db magnitude corresponding to $\omega_m = -20 \log \frac{1}{\sqrt{\alpha}} = -20 \log \frac{1}{\sqrt{0.25}} = -6$ db.

From the bode plot of uncompensated system the frequency ω_m corresponding to a db gain of –6 db is found to be 5.6 rad/sec.

$\therefore \omega_m = 5.6$ rad/sec.

$$\text{Now, } T = \frac{1}{\omega_m\sqrt{\alpha}} = \frac{1}{5.6\sqrt{0.25}} = 0.357 \approx 0.36$$

Transfer function of the lead compensator

$$G_c(s) = \frac{s+\frac{1}{T}}{s+\frac{1}{\alpha T}} = \alpha\frac{(1+sT)}{(1+s\alpha T)} = 0.25\frac{(1+0.36s)}{(1+0.09s)}$$

Step-6 : Open loop transfer function of compensated system.

The block diagram of the lead compensated system is shown in fig 1.5.3.

The compensator will provide an attenuation of α. To compensate for that, an amplifier of gain $1/\alpha$ is introduced in series with compensator.

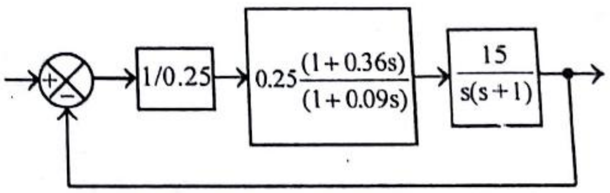


Fig 1.5.3 : Block diagram of lead compensated system

Open loop transfer function of compensated system

$$G_0(s) = \frac{1}{0.25} \times \frac{0.25\,(1+0.36s)}{(1+0.09s)} \times \frac{15}{s\,(s+1)}$$

$$= \frac{15\,(1+0.36s)}{s\,(1+0.09s)\,(1+s)}$$

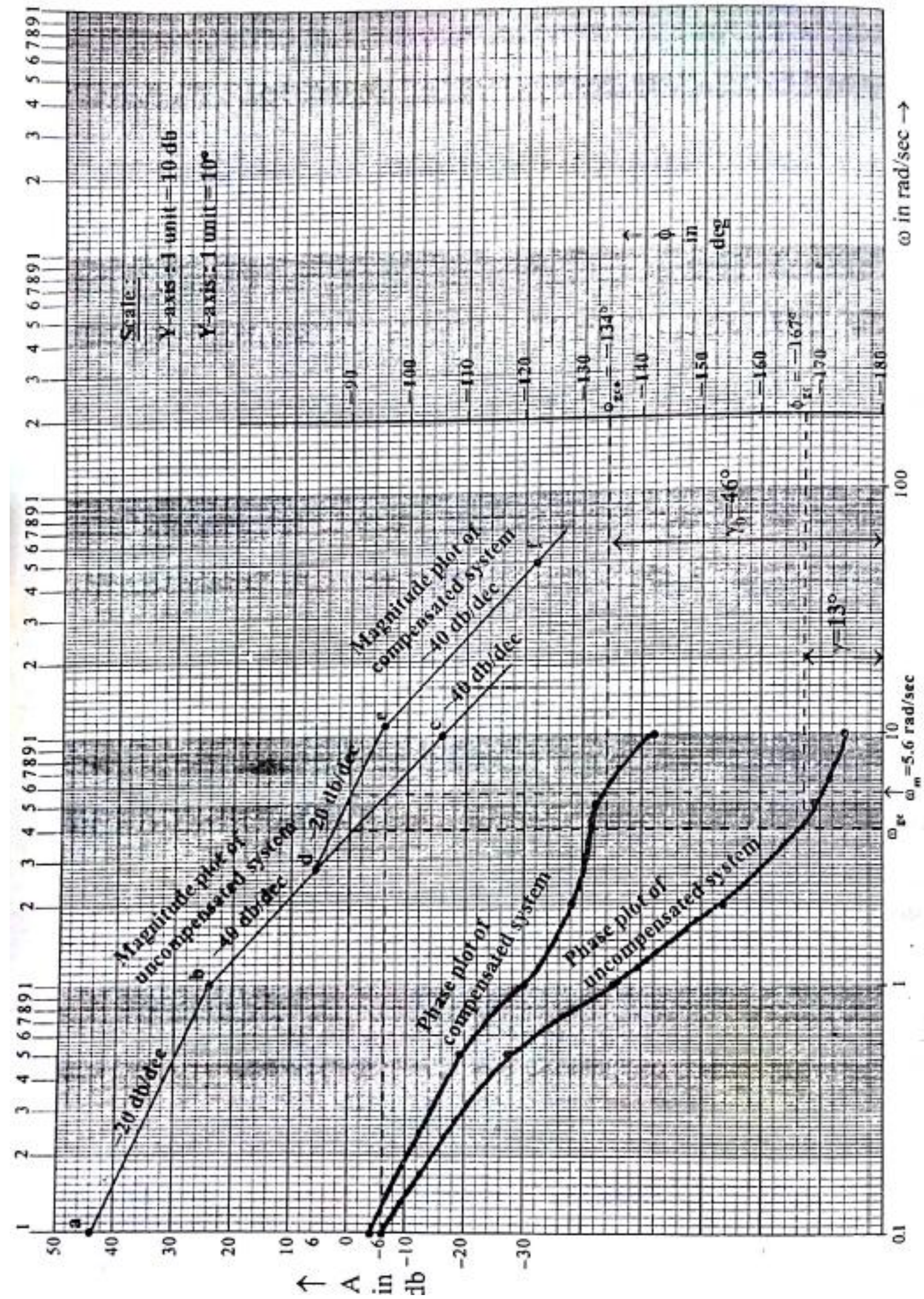
Step-7 : Draw the bode plot of compensated system to verify the design.

Put $s = j\omega$ in $G_0(s)$, $\therefore$ $G_0(j\omega) = \dfrac{15\,(1+j0.36\omega)}{j\omega\,(1+j0.09\omega)\,(1+j\omega)}$

Magnitude plot

The corner frequencies are ω_{c1}, ω_{c2} and ω_{c3}.

$\omega_{c1} = \frac{1}{1} = 1$ rad / sec ; $\omega_{c2} = \frac{1}{0.36} = 2.8$ rad / sec ; $\omega_{c3} = \frac{1}{0.09} = 11.1$ rad / sec


Scale :
Y-axis : 1 unit = 10 db
Y-axis : 1 unit = 10°
Magnitude plot of uncompensated system
Magnitude plot of compensated system
Phase plot of compensated system
Phase plot of uncompensated system
–20 db/dec
–40 db/dec
$\gamma_n = 46°$
$\gamma = 13°$
$\phi_{gc} = -167°$
$\omega_m = 5.6$ rad/sec
ω in rad/sec →
A in db
φ in deg

The various terms of $G_0(j\omega)$ are listed in table-3. Also the table shows the slope contributed by each term and the change in slope at the corner frequency.

TABLE-3

Term	Corner frequency rad/sec	Slope db/dec	Change in slope db/dec
$\frac{15}{j\omega}$	-	−20	-
$\frac{1}{1+j\omega}$	$\omega_{c1} = 1$	−20	−20 −20 = −40
$1+j0.36\omega$	$\omega_{c2} = \frac{1}{0.36} = 2.8$	+20	−40+20 = −20
$\frac{1}{1+j0.09\omega}$	$\omega_{c3} = \frac{1}{0.09} = 11.1$	−20	−20−20 = −40

Choose a low frequency ω_l such that $\omega_l < \omega_{c1}$ and choose a high frequency ω_h such that $\omega_h > \omega_{c3}$.

Let $\omega_l = 0.1$ rad/sec and $\omega_h = 50$ rad/sec

Let $A_0 = |G_0(j\omega)|$ in db

At $\omega = \omega_l = 0.1$ rad / sec, $A_0 = 20\log\left|\frac{15}{j\omega}\right| = 20\log\frac{15}{0.1} = 43.5 \text{ db} \approx 44 \text{ db}$

At $\omega = \omega_{c1} = 1$ rad / sec, $A_0 = 20\log\left|\frac{15}{j\omega}\right| = 20\log\frac{15}{1} = 23.5 \text{ db} \approx 24 \text{ db}$

At $\omega = \omega_{c2} = 2.8$ rad / sec, $A_0 = \left[\text{slope from } \omega_{c1} \text{ to } \omega_{c2} \times \log\frac{\omega_{c2}}{\omega_{c1}}\right] + \left(\begin{matrix}\text{gain at}\\ \omega = \omega_{c1}\end{matrix}\right)$

$$= -40 \times \log\frac{2.8}{1} + 24 = 6 \text{ db}$$

At $\omega = \omega_{c3} = 11.1$ rad / sec, $A_0 = \left[\text{slope from } \omega_{c2} \text{ to } \omega_{c3} \times \log\frac{\omega_{c3}}{\omega_{c2}}\right] + \left(\begin{matrix}\text{gain at}\\ \omega = \omega_{c2}\end{matrix}\right)$

$$= -20 \times \log\frac{11.1}{2.8} + 6 = -6 \text{ db}$$

At $\omega = \omega_h = 50$ rad / sec, $A_0 = \left[\text{slope from } \omega_{c3} \text{ to } \omega_h \times \log\frac{\omega_h}{\omega_{c3}}\right] + \left(\begin{matrix}\text{gain at}\\ \omega = \omega_{c3}\end{matrix}\right)$

$$= -40 \times \log\frac{50}{11.1} + (-6) = -32 \text{ db}$$

Let the points a, b, d, e and f be the points corresponding to frequencies ω_l, ω_{c1}, ω_{c2}, ω_{c3} and ω_h respectively on the magnitude plot of compensated system. The magnitude plot of compensated system is drawn on the same semilog graph sheet by using the same scales as shown in fig 1.5.2.

Phase Plot

The phase angle of $G_0(j\omega)$ as a function of ω is given by

$$\phi_0 = \angle G_0(j\omega) = \tan^{-1}0.36\omega - 90^\circ - \tan^{-1}0.09\omega - \tan^{-1}\omega.$$

The phase angle of $G_0(j\omega)$ are calculated for various values of ω and listed in table- 4.

TABLE-4

ω rad/sec	0.1	0.5	1	2	5	10	
ϕ_0 deg		−94	−109	−120	−128	−132	−142

In the same semilog sheet and by using the same scales, the phase plot of compensated system is sketched as shown in fig 1.5.2.

Let, ϕ_{gc0} = Phase of $G_0(j\omega)$ at new gain crossover frequency.

and γ_0 = Phase margin of compensated system.

From the bode plot of compensated system we get, $\phi_{gc0} = -134^\circ$.

Now, $\gamma_0 = 180^\circ + \phi_{gc0} = 180^\circ - 134^\circ = 46^\circ$

CONCLUSION

The phase margin of the compensated system is satisfactory. Hence the design is acceptable.

RESULT

The transfer function of lead compensator, $G_c(s) = \dfrac{0.25\,(1+0.36s)}{(1+0.09s)} = \dfrac{(s+2.78)}{(s+11.11)}$

$$\left.\begin{array}{l}\text{Open loop transfer function}\\ \text{of lead compensated system}\end{array}\right\} G_0(s) = \frac{15(1+0.36s)}{s\,(1+0.09s)\,(1+s)}$$

Lead compensator	**Lag** compensator
oHigh pass	oLow pass

oApproximates derivative plus proportional control	oApproximates integral plus proportional control
oContributes phase lead	oAttenuation at high frequencies
oIncreases the gain crossover frequency	oMoves the gain-crossover frequency lower
oIncreases bandwidth	oReduces bandwidth

BEE003 Advanced Control System
Unit -IV
PART –A (2 MARKS)

1. **How nonlinearities are classified ?**
   They are
   - Incidental
   - Intentional

   The incidental nonlinearities are those which are inherently present in the system. Eg. Saturation, dead-zone, couloumb friction, stiction, backlash, etc.
   The intentional nonlinearities are those which are deliberately inserted in the system to modify the system characteristics. Eg. Relay.

2. **What are the linear and nonlinear systems ?**
   The linear systems are system in which obeys the principles of superposition. The system which does not satisfy superposition principle is called nonlinear system.
   Linear systems are those where principle of superposition (if the two inputs are applied simultaneously, then output will be the sum of two outputs) is applicable but in case of highly non linear system we cannot apply principle of superposition

3. **What are limit cycles ?**
   The limit cycles are oscillations of the response (or output) of nonlinear systems with fixed amplitude and frequency. If these oscillations or limit cycles exists when there is no input then they are called zero input limit cycles.

4. **What is saturation?**
   In saturation non linearity, the output is proportional to input for a limited range of input signals. When input exceeds this range, the output tends to become nearly constant.
   Eg. Output of electronic, rotating and flow amplifiers

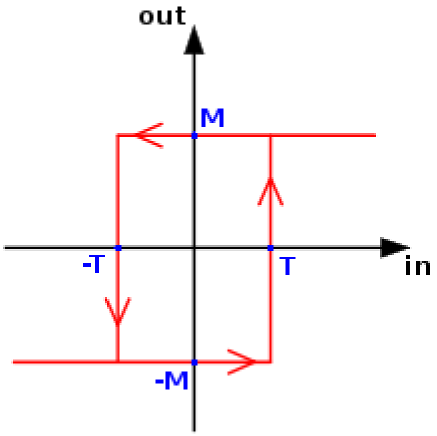
5. **What is dead zone?**

   A dead-zone is a kind of non linearity in which the system doesn't respond to the given input until the input reaches a particular level. The dead-zone is the region in which the output is zero for a given input. When the input is increased beyond this dead-zone value, the output will be linear.

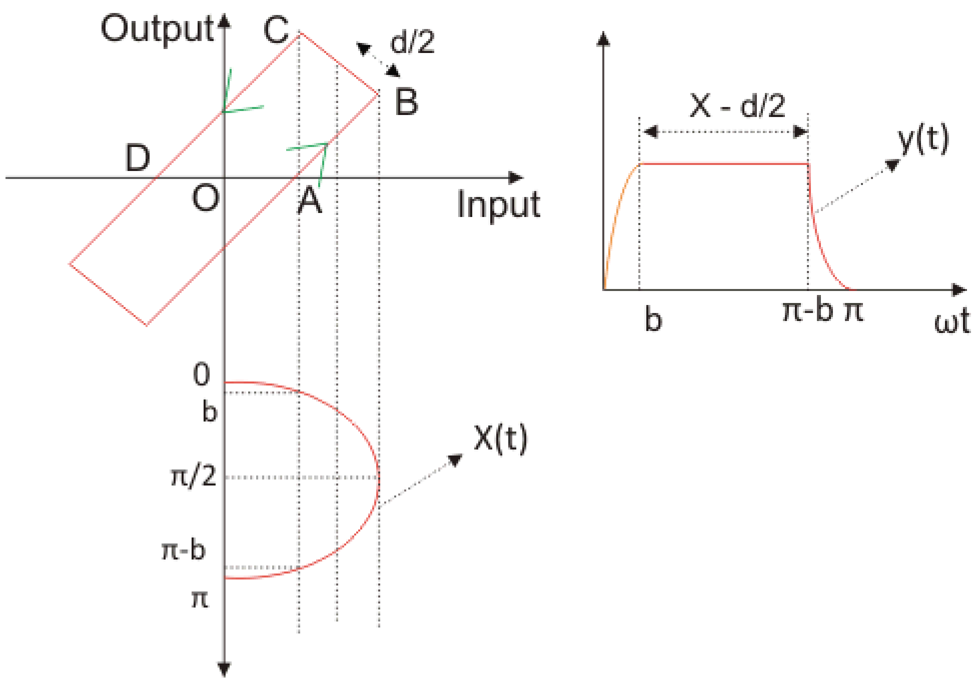
6. **What is hysteresis and backlash ?**

The **hysteresis** is a phenomenon in which the output follows a different path for increasing and decreasing values of input.

The **backlash nonlinearity** is a type of hysteresis in mechanical gear trains and linkages.



Input-Output characteristics of relay with hysteresis



Input-Output characteristics of backlash nonlinearity

7. What is describing function ?

When the input, x to the nonlinearity is a sinusoidal signal [i.e. x= X sin t], the describing function of the nonlinearity is defined as,

Describing function, K (X,) =

Where , Y = Amplitude of the functional harmonic component of the output

O = Phase shift of the functional harmonic component of the output

With respect to the input.

X = Maximum value of input signal

= Angular frequency of input signal

8. What is phase plane ?

The coordinate plane with the state variables x and x as two axes is called the phase plane [i.e. in phase plane x is represented in x-axis, and x in y-axis].

9. **What is phase trajectory ?**

The locus of the state point [x , x] in phase plane with time as running parameter is called phase trajectory.

10. **What is root locus technique ?**

Root locus technique in control system used for determining the stability of the given system. Root locus is a graphical representation of the closed-loop poles as a system parameter is varied .It can be used to describe qualitatively the performance of a system as various parameters are changed It gives graphic representation of a system's transient response and also stability We can see the range of stability, instability, and the conditions that cause a system to break into oscillation

PART –B (6 MARKS)

1. **Discuss the characteristics of non linear system.**

The characteristics of non linear system are given below :

1. The response of nonlinear system to a particular test signal is no guide to their behaviour to other inputs, since the principle of superposition does not holds good for nonlinear systems.
2. The nonlinear system response may be highly sensitive to input amplitude. The stability study of nonlinear systems requires the information about the type and amplitude of the anticipated inputs, initial conditions, etc., in addition to the usual requirement of the mathematical model.
3. The nonlinear systems may exhibit limit cycles which are self sustained oscillations of fixed frequency and amplitude.
4. The non-linear systems may have jump resonance in the frequency response.
5. The output of a nonlinear system will have harmonics and sub-harmonics when excited by sinusoidal signals.

2. Write the describing function of backlash nonlinearity and its characteristics.

Write the describing function of backlash non-linearity.

When the input, $x = X \sin\omega t$, the describing function of backlash non-linearity whose input-output characteristic is shown in fig Q2.24 is given by

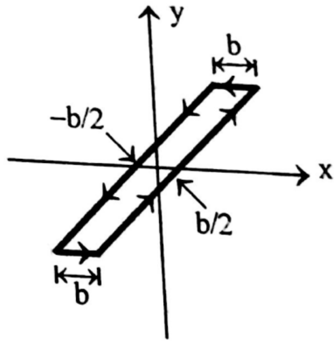


Fig Q2.24 : Input-Output characteristic of backlash nonlinearity

$$K_N(X, \omega) = \frac{Y_1}{X} \angle \phi_1$$

where, $$Y_1 = \frac{X}{\pi}\left[\left((\pi/2) + \beta + (1/2)\sin 2\beta\right)^2 + \cos^4\beta\right]^{\frac{1}{2}}$$

$$\phi_1 = \tan^{-1}\left(\frac{-\cos^2\beta}{(\pi/2) + \beta + (1/2)\sin 2\beta}\right) \qquad \therefore \beta = \sin^{-1}\left(1 - \frac{b}{X}\right)$$

3. How the stability of nonlinear system is determined using describing function.

In the stability analysis, let us assume that the linear part of the system is stable. To determine the stability of the system due to non-linearity sketch the $-1/K_N$ locus and $G(j\omega)$ locus (polar plot of $G(j\omega)$) in complex plane, (Use either a polar graph sheet or ordinary graph sheet) and from the sketches the following conclusions can be obtained.

1. If the $-1/K_N$ locus is not enclosed by the $G(j\omega)$ locus then the system is stable or there is no limit cycle at steady state.
2. If the $-1/K_N$ locus is enclosed by the $G(j\omega)$ locus then the system is unstable.
3. If the $-1/K_N$ locus and the $G(j\omega)$ locus intersect, then the system output may exhibit a sustained oscillation or a limit cycle. The amplitude of the limit cycle is given by the value of $-1/K_N$ locus at the intersection point. The frequency of the limit cycle is given by the frequency of $G(j\omega)$ corresponding to the intersection point.

4. Determine the describing function for the non-linearity system shown in figure

Solution : Input $x = X \sin\omega t$

Output can be expressed as Fourier series

$$y(t) = A_0 + \sum_{n=1}^{\infty} (A_n \cos n\omega t + B_n \sin n\omega t)$$

For symmetrical non-linearity $A_0 = 0$

$$y(t) = M + Kx \sin\omega t$$

Since, $y(t)$ is an odd function, its Fourier series has only sine terms

∴ Fundamental harmonic components is

$$y(t) = B_1 \sin\omega t$$

$$B_1 = \frac{1}{\pi}\int_0^{2\pi} y(t) \sin \omega t \, d\omega t = \frac{4}{\pi}\int_0^{\pi/2} (M + Kx \sin \omega t) \sin \omega t \, d\omega t$$

$$= \frac{4}{\pi}\int_0^{\pi/2} (M \sin \omega t + Kx \sin^2 \omega t) \, d\omega t = \frac{4}{\pi}\int_0^{\pi/2} \left[\left(M \sin \omega t + \frac{Kx}{2}(1 - \cos 2\omega t\right)\right] d\omega t$$

$$B_1 = \frac{4M}{\pi} + Kx$$

$$\therefore \quad G_D(j\omega) = \frac{\sqrt{A_1^2 + B_1^2}}{x}$$

$$\therefore \quad \boxed{G_D(j\omega) = \frac{4M}{\pi x} + K} \quad \textbf{Ans.}$$

5. Write the describing function of relay with dead zone and relay with hysteresis.

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Unit -IV

Write the describing function of relay with dead-zone.

When the input, $x = X \sin\omega t$, the describing function of relay with dead-zone whose input-output characteristics is shown in fig Q2.22 is given by

$$K_N(X, \omega) = \begin{cases} 0 & ; X < \dfrac{D}{2} \\ \dfrac{4M}{\pi X}\sqrt{1-\left(\dfrac{D}{2X}\right)^2} & ; X > \dfrac{D}{2} \end{cases}$$

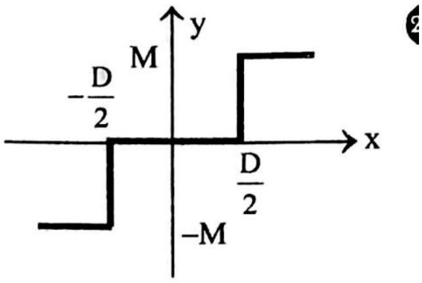


Fig Q2.22 : Input - Output characteristics of relay with dead-zone

Write the describing function of relay with hysteresis.

When the input, $x = X \sin\omega t$, the describing function of relay with hysteresis whose input-output characteristics is shown in fig Q2.23 is given by

$$K_N(X, \omega) = \begin{cases} 0 & ; \text{ when } X < \dfrac{H}{2} \\ \dfrac{4M}{\pi X} \angle\left(-\sin^{-1}\dfrac{H}{2X}\right) & ; \text{ when } X > \dfrac{H}{2} \end{cases}$$

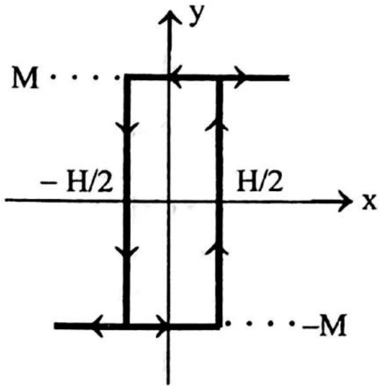
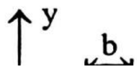


Fig Q2.23 : Input - Output characteristics of relay with hysteresis



PART –C (10 MARKS)

1. **Discuss about saturation and dead zone.**

What is saturation? Give an example.

In saturation non-linearity the output is proportional to input for a limited range of input signals. When the input exceeds this range, the output tends to become nearly constant as shown in fig Q2.12.

Saturation in the output of electronic, rotating and flow amplifiers, speed and torque saturation in electric and hydraulic motors are examples of saturation.

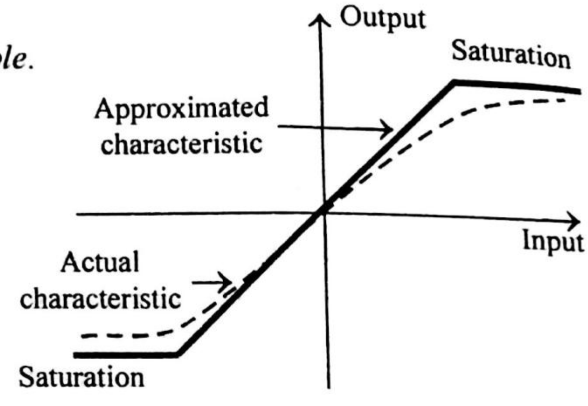


Fig Q2.12 : Saturation non-linearity

What is dead-zone?

The dead-zone is the region in which the output is zero for a given input. When the input is increased beyond this dead-zone value, the output will be linear.

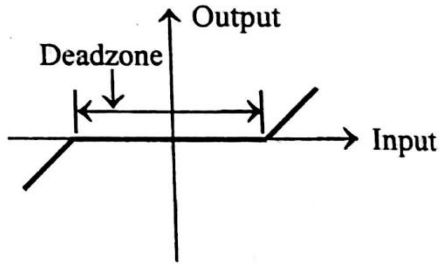


Fig Q2.13 : Dead zone nonlinearity

2. Explain how to replace nonlinearity can be replaced by describing function with block diagram.

Consider the block diagram of the nonlinear system shown in figure 2.8.

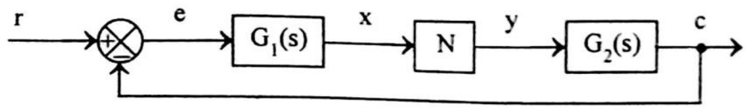


Fig 2.8 : A nonlinear system

In the above system the blocks $G_1(s)$ and $G_2(s)$ represents linear elements and the block N represent nonlinear element.

Let $x = X \sin\omega t$ be the input to nonlinear element. Now the output y of the nonlinear element will be in general a nonsinusoidal periodic function. The fourier series representation of the output y can be expressed as (by assuming that the nonlinearity does not generate subharmonics).

$$y = A_0 + A_1 \sin\omega t + B_1 \cos\omega t + A_2 \sin 2\omega t + B_2 \cos 2\omega t + \ldots\ldots \qquad \ldots\ldots(2.1)$$

If the nonlinearity is symmetrical the average value of y is zero and hence the output y is given by

$$y = A_1 \sin\omega t + B_1 \cos\omega t + A_2 \sin 2\omega t + B_2 \cos 2\omega t + \ldots\ldots \quad \ldots..(2.2)$$

In the absence of an external input (i.e, when r = 0)the output y of the nonlinearity N is fedback to its input through the linear elements $G_2(s)$ and $G_1(s)$ in tandem. If $G_1(s)G_2(s)$ has low-pass characteristics, then all the harmonics of y are filtered, so that the input x to the nonlinear element N is mainly contributed by the fundamental component of y and hence x remains sinusoidal. Under such conditions the harmonics of the output are neglected and the fundamental component of y alone considered for the purpose of analysis.

$$\therefore\ y = y_1 = A_1 \sin\omega t + B_1 \cos\omega t = Y_1 \angle\phi_1 = Y_1 \sin(\omega t + \phi_1) \quad \ldots..(2.3)$$

where, $Y_1 = \sqrt{A_1^2 + B_1^2}$ .....(2.4)

and $\phi_1 = \tan^{-1}\dfrac{B_1}{A_1}$ .....(2.5)

Y_1 = Amplitude of the fundamental harmonic component of the output.

ϕ_1 = Phase shift of the fundamental harmonic component of the output with respect to the input.

The coefficients A_1 and B_1 of the fourier series are given by

$$A_1 = \frac{2}{2\pi}\int_0^{2\pi} y \sin\omega t \ d(\omega t) \quad \ldots..(2.6)$$

$$B_1 = \frac{2}{2\pi}\int_0^{2\pi} y \cos\omega t \ d(\omega t) \quad \ldots...(2.7)$$

When the input, x to the nonlinearity is sinusoidal (i.e., x = Xsinωt) the describing function of the nonlinearity is defined as,

$$K_N(X,\omega) = \frac{Y_1}{X}\angle\phi_1 \quad \ldots..(2.81)$$

The nonlinear element N in the system can be replaced by the describing function as shown in figure 2.9.

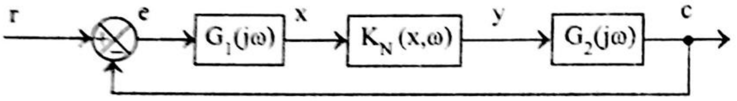


Fig 2.9 : Nonlinear system with nonlinearity replaced by describing function

If the nonlinearity is replaced by a describing function then all linear theory frequency domain techniques can be used for the analysis of the system. The describing functions are used only for stability analysis and it is not directly applied to the optimization of system design. The describing function is a frequency domain approach and no general correlation is possible between time and frequency responses.

3. Discuss the procedure for the construction of phase trajectory using isoclines method.

CONSTRUCTION OF PHASE TRAJECTORY BY ISOCLINE METHOD

Let, S = Slope at any point in the phase-plane.

From equ(2.120) we get,

$$S = \frac{dx_2}{dx_1} = \frac{f_2(x_1, x_2)}{f_1(x_1, x_2)} \quad .....(2.121)$$

Let, S_1 = Slope at a point on phase trajectory-1.

From equ(2.121) we get,

$$f_2(x_1, x_2) = S_1 \times f(x_1, x_2) \quad .....(2.122)$$

The equ(2.122) defines the locus of all such points in phase-plane at which the slope of the phase-trajectory is S_1. A locus passing through the points of same slope in

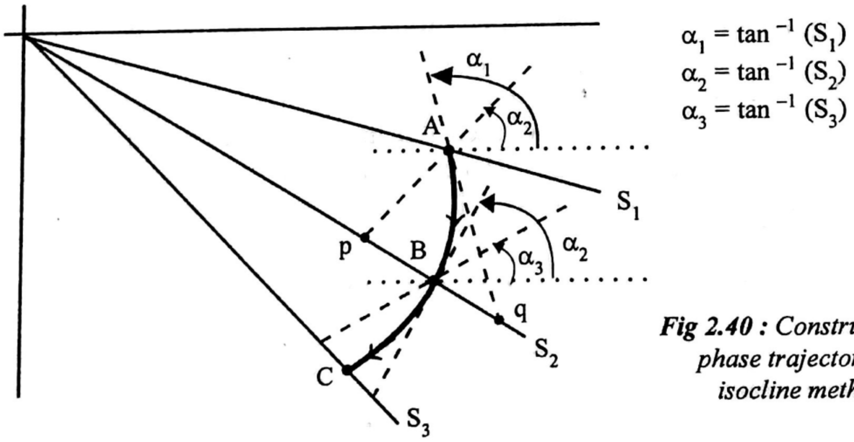


Fig 2.40 : Construction of phase trajectory by isocline method

phase-plane is called isocline. The slope of a phase trajectory at the crossing point of an isocline will be the slope of corresponding isocline. a typical plot of isoclines for various values of slope, S are shown in fig 2.40. Using these isoclines the phase trajectories can be constructed as explained below.

The phase trajectory start at a point corresponding to initial conditions. (For each set of initial conditions one phase trajectory can be constructed).

Let, S_1, S_2, S_3, etc., be the slopes associated with isoclines 1, 2, 3, etc.,

Let, $\alpha_1 = \tan^{-1}(S_1)$; $\alpha_2 = \tan^{-1}(S_2)$; $\alpha_3 = \tan^{-1}(S_3)$; etc.,

Note : *If a straight line is drawn at an angle α from a point, then the slope of the line at that point is tan α .*

In fig 2.40, let point A on isocline-1 be the point corresponding to a set of initial conditions. The phase-trajectory will leave the point A at a slope S_1. When the trajectory reaches the isocline-2, the slope changes to S_2.

Draw two lines from point A one at a slope of S_1 (i.e., at angle of $\alpha_1 = \tan^{-1}(S_1)$ and the other at a slope of S_2 (i.e., at angle of $\alpha_2 = \tan^{-1}(S_2)$. Let these two lines meet the isocline-2 at p and q. Now we can say that the trajectory would cross the iscoline-2 at a point midway between p and q. Mark the point B on the isocline-2 approximately midway between p and q.

The constructional procedure is now repeated at B to find the crossing point C on the isocline-3. By similar procedure the crossing points on the isolines are determined. A smooth curve drawn through the crossing points gives the phase-trajectory starting at point A.

The accuracy of the trajectory is closely related to the spacing of the isoclines. The phase trajectory will be more accurate if large number of isoclines are used which are very close to each other. It should be noted that using a set of isoclines, any number of trajectories can be constructed.

[OR]

7.3.1 Isoclines Method:

Let the state equations for a nonlinear system be in the form

$$\dot{x}_1 = f_1(x_1, x_2)$$
$$\dot{x}_2 = f_2(x_1, x_2)$$

Where both $f_1(x_1, x_2)$ and $f_2(x_1, x_2)$ are analytic.

From the above equation, the slope of the trajectory is given by

$$\frac{dx_2}{dx_1} = \frac{f_2(x_1, x_2)}{f_1(x_1, x_2)} = M$$

Therefore, the locus of constant slope of the trajectory is given by

$$f_2(x_1, x_2) = M f_1(x_1, x_2)$$

The above equation gives the equation to the family of isoclines. For different values of M, the slope of the trajectory, different isoclines can be drawn in the phase plane. Knowing the value of M on a given isoclines, it is easy to draw line segments on each of these isoclines.

Consider a simple linear system with state equations

$\dot{x}_1 = x_2$

$$\dot{x}_2 = -x_2 - x_1$$

Dividing the above equations we get the slope of the state trajectory in the x_1-x_2 plane as

$$\frac{dx_2}{dx_1} = \frac{-x_2 - x_1}{x_2} = M$$

For a constant value of this slope say M, we get a set of equations

$$x_2 = \frac{-1}{(M+1)} x_1$$

which is a straight line in the x_1-x_2 plane. We can draw different lines in the x_1-x_2 plane for different values of M: called isoclines. If draw sufficiently large number of isoclines to cover the complete state space as shown, we can see how the state trajectories are moving in the state plane. Different trajectories can be drawn from different initial conditions. A large number of such trajectories together form a phase portrait. A few typical trajectories are shown in figure given below.

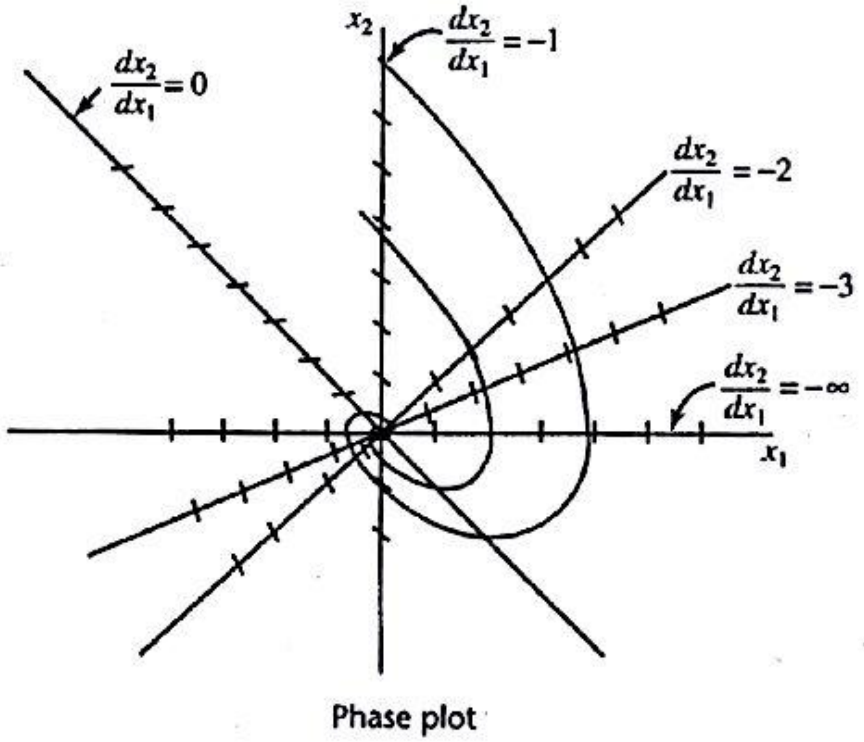


Figure 7.1

The Procedure for construction of the phase trajectories can be summarised as below:

1. For the given nonlinear differential equation, define the state variables as x_1 and x_2 and obtain the state equations as

$$\dot{x}_1 = x_2 \qquad \dot{x}_2 = f(x_1, x_2)$$

2. Determine the equation to the isoclines as

$$\frac{dx_2}{dx_1} = \frac{f(x_1, x_2)}{x_2} = M$$

3. For typical values of M. draw a large number of isoclines in x_1-x_2 plane
4. On each of the isoclines. draw small line segments with a slope M.
5. From an initial condition point, draw a trajectory following the line segments with slopes M on each of the isoclines.

4. Explain delta method to draw the phase trajectories.

7.3.2 Delta Method:

The delta method of constructing phase trajectories is applied to systems of the form

$$\ddot{x} + f(x, \dot{x}, t) = 0$$

Where $f(x, \dot{x}, t)$ may be linear or nonlinear and may even be time varying but must be continuous and single valued.

With the help of this method, phase trajectory for any system with step or ramp or any time varying input can be conveniently drawn. The method results in considerable time saving when a single or a few phase trajectories are required rather than a complete phase portrait.

While applying the delta method, the above equation is first converted to the form

$$\ddot{x} + \omega_n[x + \delta(x, \dot{x}, t)] = 0$$

In general, $\delta(x, \dot{x}, t)$ depends upon the variables x, $\dot{x}$ and t, but for short intervals the changes in these variables are negligible. Thus over a short interval, we have

$\ddot{x} + \omega_n[x + \delta] = 0$, where δ is a constant.

Let us choose the state variables as $x_1 = x$; $x_2 = \dot{x}/\omega_n$, then

$$\dot{x}_1 = \omega_n x_2$$

$$\dot{x}_2 = -\omega_n(x_1 + \delta)$$

Therefore, the slope equation over a short interval is given by

$$\frac{dx_2}{dx_1} = -\frac{x_1 + \delta}{x_2}$$

With δ known at any point P on the trajectory and assumed constant for a short interval, we can draw a short segment of the trajectory by using the trajectory slope dx_2/dx_1 given in the above equation. A simple geometrical construction given below can be used for this purpose.

1. From the initial point, calculate the value of δ.
2. Draw a short arc segment through the initial point with (-δ, 0) as centre, thereby determining a new point on the trajectory.
3. Repeat the process at the new point and continue.

5. **A linear second order servo is described by the equation $\ddot{e} + 2\xi\omega_n\dot{e} + \omega_n{}^2 e = 0$ where $\xi = 0.15$, $\omega_n = 1\ rad/sec$, $e(0) = 1.5$, $\dot{e}(0) = 0$Determine the singular points and slope.**

SOLUTION

Let x_1 and x_2 be the state variables of the system and they are related to the system variable, e as shown below.

$$x_1 = e \quad .....(2.5.1)$$

$$x_2 = \dot{e} \quad .....(2.5.2)$$

On differentiating equ(2.5.1) we get,

$$\dot{x}_1 = \dot{e}_1 = x_2 \quad .....(2.5.3)$$

On differentiating equ(2.5.2) we get,

$$\dot{x}_2 = \ddot{e} \quad .....(2.5.4)$$

Given that, $\ddot{e} + 2\zeta\omega_n\dot{e} + \omega_n^2 e = 0$ .....(2.5.5)

On substituting equation (2.5.1), (2.5.2) and (2.5.4) in equ (2.5.5) we get,

$$\dot{x}_2 + 2\zeta\omega_n x_2 + \omega_n^2 x_1 = 0$$

$$\therefore \dot{x}_2 = -\omega_n^2 x_1 - 2\zeta\omega_n x_2 \quad .....(2.5.6)$$

The state equations of the system are given by equations (2.5.3) and (2.5.6)

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -\omega_n^2 x_1 - 2\zeta\omega_n x_2$$

The singular point is obtained from state equations by putting $\dot{x}_1 = 0$ and $\dot{x}_2 = 0$

Let the coordinates of singular point in phase plane $= (x_1^0, x_2^0)$

On substituting $\dot{x}_1 = 0$ and $x_2 = x_2^0$ in equ(2.5.3) we get, $x_2^0 = 0$.

On substituting $\dot{x}_2 = 0$, $x_1 = x_1^0$ and $x_2 = x_2^0$ in equ(2.5.6) we get,

$$0 = -\omega_n^2 x_1^0 - 2\zeta\omega_n^2 x_2^0$$

But $x_2^0 = 0$, $\therefore 0 = -\omega_n^2 x_1^0$ (or) $x_1^0 = 0$

Therefore, the coordinates of singular point are (0, 0) and so the origin is the singular point.

The slope of the phase trajectory is given by

$$S = \frac{dx_2/dt}{dx_1/dt} = \frac{\dot{x}_2}{\dot{x}_1} \quad .....(2.5.7)$$

On substituting for $\dot{x}_1$ and $\dot{x}_2$ from equations (2.5.3) and (2.5.6) in equ(2.5.7) we get,

$$S = -\frac{(\omega_n^2 x_1 + 2\zeta\omega_n x_2)}{x_2} \quad .....(2.5.8)$$

Put $\zeta = 0.15$ and $\omega_n = 1$, in equ(2.5.8)

$$S = \frac{-(x_1 + 2 \times 0.15\ x_2)}{x_2} = \frac{-(x_1 + 0.3\ x_2)}{x_2}$$

$$= \frac{-x_1}{x_2} - \frac{0.3x_2}{x_2} = \frac{-x_1}{x_2} - 0.3$$

$$\therefore \frac{x_1}{x_2} = -0.3 - S \qquad \text{(or)} \qquad \frac{x_2}{x_1} = \frac{1}{-0.3 - S}$$

$$\therefore x_2 = \frac{x_1}{-0.3 - S}$$

BEE003 Advanced Control System
Unit -V
PART –A (2 MARKS)

1. **What is absolute stability?**
   If a system output is stable for all variations of its parameters, then the system is called absolute stability.

2. **What is BIBO Stability?**
   The requirement of the BIBO stability is that the absolute integral of the impulse response of the system should take only the finite value.

3. **Define Asymptotic Stability.**
   A system is asymptotically stable, if in the absence of input , the output tends towards zero irrespective of intial conditions.

4. **What is Routh stability criterion?**
   Routh criterion states that the necessary and sufficient condition for stability is that all of the elements in the first column of the routh array is positive. If this condition is not met, the system is unstable and the number of sign changes in the elements of the first column of routh array corresponds to the number of roots of characteristic equation in the right half of the S-plane.

5. **What is the principle of Lyapunov Stability Theorem.**
   A scalar function V(x) which for some real numbers $\varepsilon > 0$ satisfying the following properties for all x in the region.
   $||X|| < \varepsilon$:-
   a) $V(x) > 0; x \neq 0$
   b) $V(0) = 0$ } **ie; V(x) is Positive defined function**
   c) V(x) has continuous partial derivative with respect to all components of x
   (i) Stable: If the deroivative $dv/dt \leq 0$, dv is a negative semidefinite scalar.

6. **List the time domain specifications.**
   The time domain specifications are:
   i.Delay time
   ii. Rise time
   iii. Peak time
   iv. Peak overshoot
   v. Settling time

7. **What is an order of a system?**

The order of the system is given by the order of the differential equation governing the system. It is also given by the maximum power of s in the denominator polynomial of transfer function. The maximum power of s also gives the number of poles of the system and so the order of the system is also given by number of poles of the transfer function.

8. Define Relative stability.

Relative stability is the degree of closeness of the system; it is an indication of strength or degree of stability.

9. What is necessary and sufficient condition for stability by using R-H criterion?

The necessary and sufficient condition for stability is that all of the elements in the first column of the routh array is positive.

10. What is limitedly stable system?

For a bounded input signal, if the output has constant amplitude oscillations , then the system may be stable or unstable under some limited constraints. Such a system is called limitedly stable system.

PART –B (6 MARKS)

1. Determine the stability of the system.

Solution : Given that

$$A = \begin{bmatrix} -1 & -2 \\ 1 & -4 \end{bmatrix}$$

$$A^T = \begin{bmatrix} -1 & 1 \\ -2 & 4 \end{bmatrix}$$

Select $Q = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

we know that $Q = A^T P + PA$

$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} + \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}\begin{bmatrix} -1 & -2 \\ 1 & -4 \end{bmatrix} \quad (5.5$$

$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} -2p_{11} + 2p_{12} & -2p_{11} - 5p_{12} + p_{22} \\ -2p_{11} - 5p_{12} + p_{22} & -4p_{12} - 8p_{22} \end{bmatrix} \quad (5.5$$

Take $p_{12} = p_{21}$, this is because solution matrix P is known to be a positive definite real symme matrix for a stable system

$$-2p_{11} + 2p_{12} = -1 \quad (5.6$$

$$-2p_{11} - 5p_{12} + p_{22} = 0 \quad (5.6$$

$$-4p_{12} - 8p_{22} = -1 \quad (5.6$$

Solving the equation 5.60 and 5.61 we get

$$7p_{12} - p_{22} = -1 \quad (5.6$$

Now solving (5.62) and (5.63) we get

$p_{12} = -7/60$, $p_{21} = -7/60$

From (5.60) we get

$p_{11} = 23/60$

From (5.62)

$p_{22} = 11/60$

$$P = \begin{bmatrix} 23/60 & -7/60 \\ -7/60 & 11/60 \end{bmatrix} = \frac{1}{60}\begin{bmatrix} 23 & -7 \\ -7 & 11 \end{bmatrix}$$

Use sylvestor's theorem.

(a) $23 = r_{11} > 0$

(b) $23 \times 11 - 49 = 204 > 0$

Hence, P is positive definite.

Therefore, the origin of the system under consideration is asymptotically stable in-the-large

2. **Examine the stability of the system described by the equation.**

Solution : Let $V(x) = x_1^2 + x_2^2$

$$\dot{V}(x) = \frac{\partial V}{\partial x_1}\dot{x}_1 + \frac{\partial V}{\partial x_2}\dot{x}_2$$

$$\dot{V}(x) = 2x_1\dot{x}_1 + 2x_2\dot{x}_2$$

Put the values of $\dot{x}_1$ and $\dot{x}_2$

$$\dot{V}(x) = 2x_1(x_2) + 2x_2(-6x_1 - 5x_2)$$

$$= -10x_1x_2 - 10x_2^2 = \text{negative definite}$$

Hence, the system is asymptotically stable.

3. **Explain the direct method of lyapunov?**

4. **What is BIBO stability and asymptotic stability?**

The requirement of the BIBO stability is that the absolute integral of the impulse response of the system should take only the finite value.

A system is asymptotically stable, if in the absence of input, the output tends towards zero irrespective of initial conditions.

5. **The system is given by**

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -x_1 - x_1^2 x_2$$

Investigate the system by Lyapunov's method using V = $x_1^2 + x_2^2$

Solution : Let $V(x) = x_1^2 + x_2^2$

$$\dot{V}(x) = \frac{\partial V}{\partial x_1}\dot{x}_1 + \frac{\partial V}{\partial x_2}\dot{x}_2$$

$$\dot{V}(x) = 2x_1\dot{x}_1 + 2x_2\dot{x}_2$$

Put the values of $\dot{x}_1$ and $\dot{x}_2$

$$\dot{V}(x) = 2x_1(x_2) + 2x_2(-6x_1 - 5x_2)$$

$$= -10x_1x_2 - 10x_2^2 = \text{negative definite}$$

Hence, the system is asymptotically stable.

PART –C (10 MARKS)

1. Explain variable gradient method to form Lyapunov function.

The Variable-Gradient Method

The quadratic form approach used so far to Lyapunov function formulation is too restrictive. Schultz and Gibson (1962) suggested the variable gradient method for generating Lyapunov function which provides considerable flexibility in selecting a suitable function.

For the autonomous system (8.45), let $V(x)$ be a candidate for a Lyapunov function. The time derivative of V can be expressed as

$$\dot{V}(\mathbf{x}) = \frac{\partial V}{\partial x_1}\dot{x}_1 + \frac{\partial V}{\partial x_2}\dot{x}_2 + \dots + \frac{\partial V}{\partial x_n}\dot{x}_n \tag{8.58}$$

Let (refer Appendix III)

$$\mathbf{g}(\mathbf{x}) = \text{grad } V(\mathbf{x}) = \begin{bmatrix} \frac{\partial V}{\partial x_1} \\ \frac{\partial V}{\partial x_2} \\ \vdots \\ \frac{\partial V}{\partial x_n} \end{bmatrix} = \begin{bmatrix} g_1(\mathbf{x}) \\ g_2(\mathbf{x}) \\ \vdots \\ g_n(\mathbf{x}) \end{bmatrix} \tag{8.59}$$

Then eqn. (8.58) may be written as

$$\dot{V}(\mathbf{x}) = (\mathbf{g}(\mathbf{x}))^T \dot{\mathbf{x}} \tag{8.60}$$

The Lyapunov function can be generated by integrating (8.60) on both sides.

$$V(\mathbf{x}) = \int_0^t \frac{dV(\mathbf{x})}{dt}\, dt = \int_0^t (\mathbf{g}(\mathbf{x}))^T \frac{d\mathbf{x}}{dt}\, dt$$

$$= \int_0^{\mathbf{x}} (\mathbf{g}(\mathbf{x}))^T \, d\mathbf{x} \tag{8.61}$$

This is a line integral from origin to an arbitrary point $(x_1, x_2, \dots, x_n)$ in the state space. Since $(\mathbf{g}(\mathbf{x}))^T d\mathbf{x} = dV(\mathbf{x})$, the integral in (8.61) is independent of the path of integration. The simplest path is indicated in Fig. 8.14 for a three-dimensional system. This shows that the integral in (8.61) can be

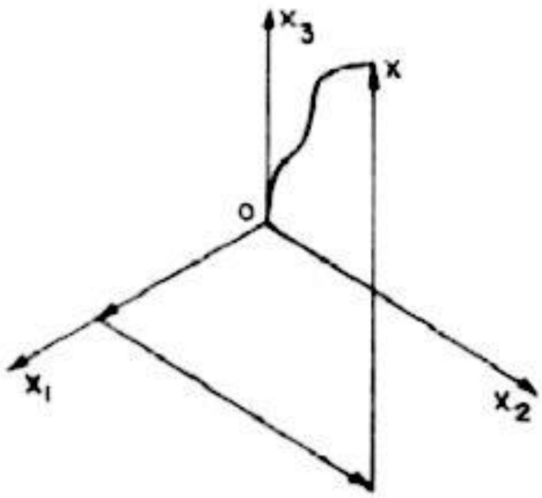


Fig. 8.14

evaluated sequentially along the component directions $\{x_1, x_2, \ldots, x_n\}$ of the state vector $\mathbf{x}$, i.e.,

$$V(\mathbf{x}) = \int_0^{\mathbf{x}} (\mathbf{g}(\mathbf{x}))^T d\mathbf{x}$$

$$= \int_0^{x_1} g_1(\theta_1, 0, 0, \ldots, 0, 0)\, d\theta_1 + \int_0^{x_2} g_2(x_1, \theta_2, 0, \ldots, 0, 0)\, d\theta_2 + \cdots$$

$$\cdots + \int_0^{x_n} g_n(x_1, x_2, \ldots, x_{n-1}, \theta_n)\, d\theta_n \qquad (8.62)$$

The variable-gradient method then consists of selecting a vector function $\mathbf{g}(\mathbf{x})$ and integrating this function as per (8.62) to obtain the scalar function $V(\mathbf{x})$. However, for a continuous vector function $\mathbf{g}(\mathbf{x})$ to be gradient of a scalar $V(\mathbf{x})$, we must have

$$\frac{\partial g_i}{\partial x_j} = \frac{\partial g_j}{\partial x_i};\quad i, j = 1, 2, \ldots, n \qquad (8.63)$$

This can be easily established. From the results of Appendix III, we can write

$$\frac{\partial^2 V(\mathbf{x})}{d\mathbf{x}^2} = \begin{bmatrix} \frac{\partial^2 V}{\partial x_1^2} & \frac{\partial^2 V}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 V}{\partial x_1 \partial x_n} \\ \frac{\partial^2 V}{\partial x_2 \partial x_1} & \frac{\partial^2 V}{\partial x_2^2} & \cdots & \frac{\partial^2 V}{\partial x_2 \partial x_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial^2 V}{\partial x_n \partial x_1} & \frac{\partial^2 V}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 V}{\partial x_n^2} \end{bmatrix}$$

Since $\frac{\partial^2 V}{\partial x_i \partial x_j} = \frac{\partial^2 V}{\partial x_j \partial x_i}$, this matrix is symmetric.

$$\frac{\partial \mathbf{g}(\mathbf{x})}{\partial \mathbf{x}} = \begin{bmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} & \cdots & \frac{\partial g_1}{\partial x_n} \\ \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} & \cdots & \frac{\partial g_2}{\partial x_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial g_n}{\partial x_1} & \frac{\partial g_n}{\partial x_2} & \cdots & \frac{\partial g_n}{\partial x_n} \end{bmatrix}$$

For $\mathbf{g}(\mathbf{x})$ to be equal to grad $V(\mathbf{x})$, this matrix must be symmetric which implies (8.63). These are total $\frac{n(n-1)}{2}$ equations.

The procedure to formulate a Lyapunov function is given in the following steps.

1. To begin with, assume a completely general form given below, for a gradient vector $\mathbf{g}(\mathbf{x})$.

$$\mathbf{g}(\mathbf{x}) = \begin{bmatrix} g_1(\mathbf{x}) \\ g_2(\mathbf{x}) \\ \vdots \\ g_n(\mathbf{x}) \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n \end{bmatrix} \tag{8.64}$$

   The a_{ij}'s are completely undetermined quantities and could be constants or functions of both state variables and t. It is convenient, however, to choose a_{nn} as a constant.

2. Form $\dot{V}$ as per equation (8.60) with $\mathbf{g}(\mathbf{x})$ as in eqn. (8.64). Choose a_{ij}'s to constrain it to be negative definite or at least negative semi-definite.
3. Determine the remaining a_{ij}'s to satisfy the equations (8.63).
4. Recheck $\dot{V}$, in case step (3) has altered its definiteness.
5. Determine V by integrating as in eqn. (8.62).
6. Determine the region of stability where V is positive definite.

2. Consider the non – linear system of equations.

Example 8.9: Consider the nonlinear system of equations

$$\dot{x}_1 = -2x_1 + x_1 x_2 \qquad (8.32)$$

$$\dot{x}_2 = -x_2 + x_1 x_2$$

Note that there are two equilibrium points: $\mathbf{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. We shall study the stability of the origin.

A candidate for a Lyapunov function is

$$V = p_{11}x_1^2 + p_{22}x_2^2;\ p_{11} > 0, p_{22} > 0$$

Then

$$\frac{dV}{dt} = 2p_{11}x_1\dot{x}_1 + 2p_{22}x_2\dot{x}_2$$

$$= 2p_{11}x_1(-2x_1 + x_1x_2) + 2p_{22}x_2(-x_2 + x_1x_2)$$

$$= 2p_{11}x_1^2(x_2 - 2) + 2p_{22}x_2^2(x_1 - 1)$$

We cannot select any values of $p_{11} > 0$ and $p_{22} > 0$ to make $\frac{dV}{dt}$ globally negative definite. Let us take $p_{11} = p_{22} = 1$ and study the resulting function

$$\frac{dV}{dt} = 2x_1^2(x_2 - 2) + 2x_2^2(x_1 - 1)$$

For asymptotic stability we require that

$$x_1^2(x_2 - 2) + x_2^2(x_1 - 1) < 0 \qquad (8.33)$$

The region of state-space where this condition is not satisfied is *possibly* the region of instability. Let us concentrate on the region of state-space where this condition is satisfied. The limiting condition for such a region is

$$x_1^2(x_2-2)+x_2^2(x_1-1)=0$$

The dividing line in the first quadrant is shown in Fig. 8.10. Further, we can easily verify that condition (8.33) is satisfied in the entire third quadrant. Information about second and fourth quadrants can be obtained on similar lines.

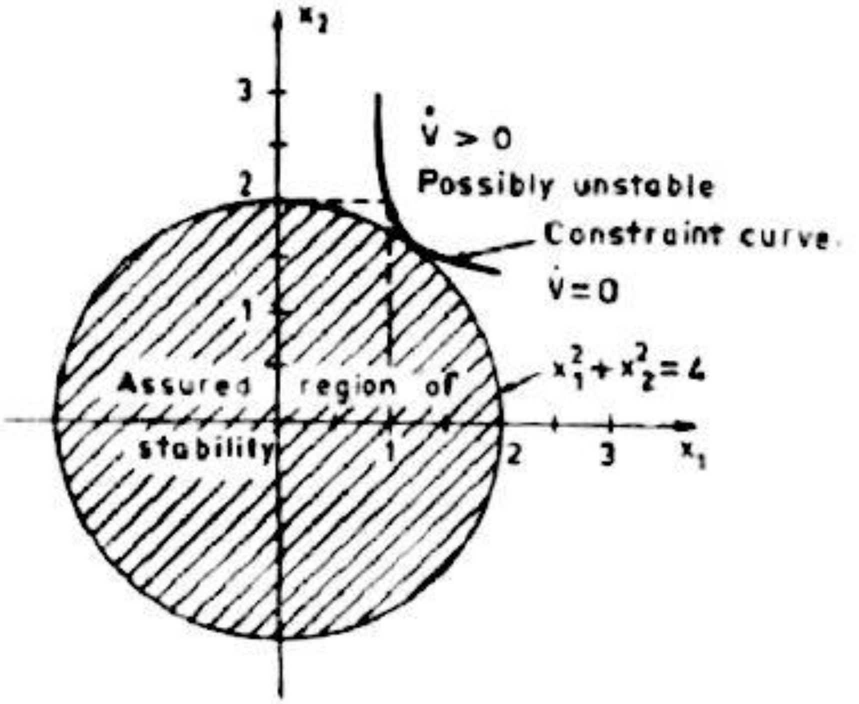


Fig. 8.10

Now, all those initial states $\mathbf{x}(0)$ for which $\mathbf{x}(t)$; $t \geq 0$ lies in the region where $\dot{V} < 0$, will lead to asymptotic stability. Let δ_k be the region (set of values of $\mathbf{x}$ including origin) in which $V(\mathbf{x})$ is positive definite, $\frac{dV}{dt}$ is negative definite and $V(\mathbf{x}) \leq k$. Then every solution with $\mathbf{x}$ in δ_k, approaches the origin asymptotically as $t \to \infty$ (refer proof of Theorem 8.10). Obviously, we would like to find the largest such region. To do this, we investigate the equation for the contour of V:

$$V = x_1^2 + x_2^2 - k$$

for various values of k, selecting that k which gives the largest region satisfying the condition $\frac{dV}{dt} < 0$. For the example under consideration, we find that $k = 4$ gives the largest region (Fig. 8.10).

3. Construct Routh – Hurwitz array and determine the stability of the system whose characteristic equation is $S^6 + 2S^5 + 8S^4 + 12S^3 + 20S^2 + 16S + 16 = 0$?

SOLUTION

The characteristic equation of the system is, $s^6+2s^5+8s^4+12s^3+20s^2+16s+16=0$.

The given characteristic polynomial is 6th order equation and so it has 6 roots. Since the highest power of s is even number, form the first row of routh array using the coefficients of even powers of s and form the second row using the coefficients of odd powers of s.

s^6	:	1	8	20	16	....Row-1
s^5	:	2	12	16		....Row-2

The elements of s^5 row can be divided by 2 to simplify the calculations.

s^6	:	1	8	20	16	....Row-1
s^5	:	1	6	8		....Row-2
s^4	:	1	6	8		....Row-3
s^3	:	0	0			....Row-4
s^3	:	1	3			....Row-4
s^2	:	3	8			....Row-5
s^1	:	0.33				....Row-6
s^0	:	8				....Row-7

↑ Column-1

$s^4 : \frac{1\times8-6\times1}{1} \quad \frac{1\times20-8\times1}{1} \quad \frac{1\times16-0\times1}{1}$

$s^4 :$ 2 12 16

divide by 2

$s^4 :$ 1 6 8

$s^3 : \frac{1\times6-6\times1}{1} \quad \frac{1\times8-8\times1}{1}$

$s^3 :$ 0 0

The auxiliary equation is, $A = s^4+6s^2+8$. On differentiating A with respect to s we get,

$$\frac{dA}{ds} = 4s^3+12s$$

The coefficients of $\frac{dA}{ds}$ are used to form s^3 row.

s^3: 4 12

divide by 4

s^3: 1 3

$s^2 : \frac{1\times6-3\times1}{1} \quad \frac{1\times8-0\times1}{1}$

$s^2 :$ 3 8

$s^1 : \frac{3\times3-8\times1}{3}$

$s^1 :$ 0.33

$s^0 : \frac{0.33\times8-0\times3}{0.33}$

$s^0 :$ 8

On examining the elements of 1st column of routh array it is observed that there is no sign change. The row with all zeros indicate the possibility of roots on imaginary axis. Hence the system is limitedly or marginally stable.

The auxiliary polynomial is,

$$s^4+6s^2+8=0$$

Let, $s^2 = x$

$$\therefore x^2+6x+8=0$$

The roots of quadratic are, $x = \frac{-6\pm\sqrt{6^2-4\times8}}{2}$
$= -3\pm1 = -2 \text{ or } -4$

The roots of auxiliary polynomial is,

$$s = \pm\sqrt{x} = \pm\sqrt{-2} \text{ and } \pm\sqrt{-4}$$
$$= +j\sqrt{2}, -j\sqrt{2}, +j2 \text{ and } -j2$$

The roots of auxiliary polynomial are also roots of characteristic equation. Hence 4 roots are lying on imaginary axis and the remaining two roots are lying on the left half of s-plane.

RESULT

1. The system is limitedly or marginally stable.
2. Four roots are lying on imaginary axis and remaining two roots are lying on left half of s-plane.

4. **Consider a non- linear system described by the equation $\dot{x_1} = -3x_1 + x_2$ $\dot{x_2}$ = -x_1 -x_2 -$x_2{}^3$ Investigate the stability of equilibrium state.**

Example 5.56. Consider a non-linear system described by the equations

$$\dot{x_1} = -3x_1 + x_2$$

$$\dot{x_2} = -x_1 - x_2 - x_2^3$$

Investigate the sbtability of the equilibrium state.

Solution : Let $V(x) = x_1^2 + x_2^2$

$$\dot{V}(x) = \frac{\partial V}{\partial x_1}\dot{x_1} + \frac{\partial V}{\partial x_2}\dot{x_2}$$

$$\dot{V}(x) = 2x_1 \dot{x}_1 + 2x_2 \dot{x}_2$$

Substitute the values of $\dot{x}_1$ and $\dot{x}_2$

$$\dot{V}(x) = 2x_1(-3x_1 + x_2) + 2x_2(-x_1 - x_2 - x_2^3)$$

$$\dot{V}(x) = -6x_1^2 - 2x_2^2 - 2x_2^4 = \text{negative definite}$$

Hence, the system is asymptotically stable in-the-large.

5. **Discuss Krasovskii's Method for stability analysis of non-linear system.**

Krasovskii's Method

Consider the system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x});\ \mathbf{f}(0) = \mathbf{0}$$ (i.e., singular point at origin)

Define a Liapunov function as

$$V = \mathbf{f}^T\mathbf{P}\mathbf{f} \qquad (14.71)$$

where **P** = a symmetric positive definite matrix.

Now

$$\dot{V} = \dot{\mathbf{f}}^T\mathbf{P}\mathbf{f} + \mathbf{f}^T\mathbf{P}\dot{\mathbf{f}} \qquad (14.72)$$

$$\dot{\mathbf{f}} = \frac{\partial \mathbf{f}}{\partial \mathbf{x}}\frac{\partial \mathbf{x}}{\partial t} = \mathbf{J}\mathbf{f}$$

where

$$\mathbf{J} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}$$ is Jacobian matrix

Substituting $\dot{\mathbf{f}}$ in (14.72), we have

$$\dot{V} = \mathbf{f}^T\mathbf{J}^T\mathbf{P}\mathbf{f} + \mathbf{f}^T\mathbf{P}\mathbf{J}\mathbf{f}$$

$$= \mathbf{f}^T(\mathbf{J}^T\mathbf{P} + \mathbf{P}\mathbf{J})\mathbf{f}$$

Let $\mathbf{Q} = \mathbf{J}^T\mathbf{P} + \mathbf{P}\mathbf{J}$

Since V is positive definite, for the system to be asymptotically stable **Q** should be negative definite. If in addition, $V(\mathbf{x}) \to \infty$ as $\|\mathbf{x}\| \to \infty$, the system is asymptotically stable in-the-large.

[OR]

BEE003 Advanced Control System
Unit -V

Krasovskii's Method:

Consider the system defined by $\dot{x} = f(x)$, where x is an n-dimensional vector. Assume that f(0) = 0 and that f(x) is differentiable with respect to x_i where, I = 1,2,3,.....,n. The Jacobian matrix F(x) for the system is

$$F(x) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \dots & \dots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \dots & \dots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & & & \vdots \\ \vdots & \vdots & & & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \dots & \dots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}$$

Define $\hat{F}(x) = F^*(x) + F(x)$, *where* $F^*(x)$ is the conjugate transpose of F(x). If the Hermitian matrix $\hat{F}(x)$ is negative definite, then the equilibrium state $x = 0$ is asymptotically stable. A Liapunov function for this system is $V(x) = f^*(x)f(x)$. If in addition $f^*(x)f(x) \rightarrow \infty$ *as* $\|x\| \rightarrow \infty$, then the equilibrium state is asymptotically stable in the large

Proof:

If $\hat{F}(x)$ is negative definite for all x ≠ 0, the determinant of $\hat{F}$ is nonzero everywhere except at $x = 0$. There is no other equilibrium state than $x = 0$ in the entire state space. Since f(0) = 0, f(x) ≠ 0 for x ≠ 0, and $V(x) = f^*(x)f(x)$. is positive definite. Note that $\dot{f}(x) = F(x)\dot{x} = F(x)f(x)$

We can obtain $\dot{V}$ as

$$\dot{V}(x) = \dot{f}^*(x)f(x) + f^*(x)\dot{f}(x)$$

$$= [F(x)f(x)]^*f(x) + f^*(x)F(x)f(x)$$
$$= f^*(x)[F^*(x) + F(x)]f(x)$$
$$= f^*(x)\hat{F}(x)f(x)$$

If $\hat{F}(x)$ is negative definite, we see that $\dot{V}(x)$ is negative definite. Hence V(x)is a Liapunov function. Therefore, the origin is asymptotically stable. If $V(x) = f^*(x)f(x)$ tends to infinity as $\|x\| \rightarrow \infty$, then the equilibrium state is asymptotically stable in the large.

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