12

Energy and Power Spectral Densities

In this chapter we study energy and power spectra and their relations to signal duration, periodicity and correlation functions.

12.1 Energy Spectral Density

Let $f(t)$ be an electric potential in Volt applied across a resistance of $R = 1$ ohm. The total energy dissipated in such a resistance is given by

$$E = \int_{-\infty}^{\infty} \left\{ f^2(t)/R \right\} dt. \tag{12.1}$$

Since the resistance value is unity the dissipated energy may be also be referred to as *normalized energy.* In what follows we shall refer to it simply as the dissipated energy, with the implicit assumption that it is the energy dissipated into a resistance of 1 ohm.

We recall Parseval's theorem which states that if a function $f(t)$ is generally complex and if $F(j\omega)$ is the Fourier transform of $f(t)$ then

$$\int_{-\infty}^{\infty} |f(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(j\omega)|^2 d\omega. \tag{12.2}$$

The energy in the resistance may therefore be written in the form

$$E = \int_{-\infty}^{\infty} f^2(t)\, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(j\omega)|^2 d\omega. \tag{12.3}$$

The function $|F(j\omega)|^2$ is called the *energy spectral density,* or simply the energy density, of $f(t)$. It is attributed the special symbol $\varepsilon_{ff}(\omega)$, that is,

$$\varepsilon_{ff}(\omega) \triangleq |F(j\omega)|^2 . \tag{12.4}$$

We note that its integral is equal to 2π times the signal energy

$$E = \frac{1}{2\pi} \int_{-\infty}^{\infty} \varepsilon_{ff}(\omega)\, d\omega \tag{12.5}$$

hence the name 'spectral density'.

Given two signals $f_1(t)$ and $f_2(t)$, where $f_1(t)$ represent a current source and $f_2(t)$ the voltage that the current source produces across a resistance R of 1 ohm, we have

$$E = \int_{-\infty}^{\infty} f_1(t) f_2(t)\, dt. \tag{12.6}$$

Parseval's or Rayleigh's theorem is written

$$\int_{-\infty}^{\infty} f_1(t)\, f_2(t)\, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} F_1(-j\omega) F_2(j\omega)\, d\omega. \tag{12.7}$$

If $f_1(t)$ and $f_2(t)$ are real

$$F_1(-j\omega) = F_1^*(j\omega), \quad F_2(-j\omega) = F^*(j\omega). \tag{12.8}$$

The *normalized cross-energy* or simply *cross-energy* is therefore given by

$$E_{f_1 f_2} = \int_{-\infty}^{\infty} f_1(t) f_2(t)\, dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} F_1^*(j\omega) F_2(j\omega)\, d\omega. \tag{12.9}$$

The function

$$\varepsilon_{f_1 f_2}(\omega) \triangleq F_1^*(j\omega) F_2(j\omega) \tag{12.10}$$

is called the *cross-energy spectral density.* The cross energy of the two signals is then given by

$$E = \frac{1}{2\pi}\int_{-\infty}^{\infty} \varepsilon_{f_1 f_2}(\omega) d\omega. \tag{12.11}$$

Example 12.1 *Consider the ideal lowpass filter frequency response shown in Fig. 12.1. We have*

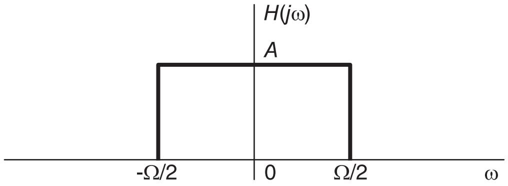


FIGURE 12.1
Ideal lowpass filter frequency response.

$$H(j\omega) = A\Pi_{\Omega/2}(\omega) = A\{u(\omega + \Omega/2) - u(\omega - \Omega/2)\}.$$

The filter's impulse response is given by

$$h(t) = \mathcal{F}^{-1}[H(j\omega)] = \frac{A\Omega}{2\pi} Sa(\Omega\, t/2).$$

The energy spectral density of $h(t)$ *is given by*

$$\varepsilon_{hh}(\omega) = |H(j\omega)|^2 = A^2\Pi_{\Omega/2}(\omega).$$

We may evaluate the energy of $h(t)$ in a finite band of frequency, say, $\Omega/4 < |\omega| < \Omega/2$, as shown in Fig. 12.2.

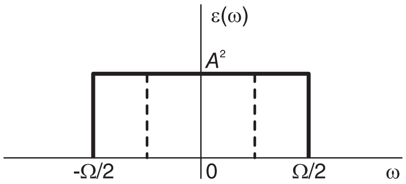


FIGURE 12.2
A frequency band of lowpass filter response.

$$E\left(\Omega/4,\ \Omega/2\right)=\frac{2}{2\pi}\int_{\Omega/4}^{\Omega/2}A^2d\omega=\frac{A^2\Omega}{4\pi}. \tag{12.12}$$

The total energy of $h\left(t\right)$ is given by

$$E=\int_{-\infty}^{\infty}h^2\left(t\right)dt=\frac{A^2}{4\pi^2}\Omega^2\int_{-\infty}^{\infty}Sa^2\left(\Omega t/2\right)dt. \tag{12.13}$$

It is easier, however, to evaluate the energy using Rayleigh's theorem. We write

$$E=\frac{2}{2\pi}\int_{0}^{\Omega/2}\varepsilon_{hh}(\omega)d\omega=\frac{A^2\Omega}{2\pi}. \tag{12.14}$$

We note that we have thus evaluated in passing the integral of the square of the sampling function. In particular, we found that

$$E=\frac{A^2\Omega^2}{4\pi^2}\int_{-\infty}^{\infty}Sa^2\left(\Omega t/2\right)dt=\frac{A^2\Omega}{2\pi}. \tag{12.15}$$

Substituting $\Omega t/2=x$, we have

$$\int_{-\infty}^{\infty}Sa^2\left(x\right)dx=\pi. \tag{12.16}$$

Example 12.2 *Let*

$$v\left(t\right)=A\cos\omega_c t$$

and

$$v_T\left(t\right)=v\left(t\right)\,\Pi_{T/2}\left(t\right)=v\left(t\right)\left\{u\left(t+T/2\right)-u\left(t-T/2\right)\right\}.$$

Evaluate the energy spectral density of this truncated sinusoid shown in Fig. 12.3.

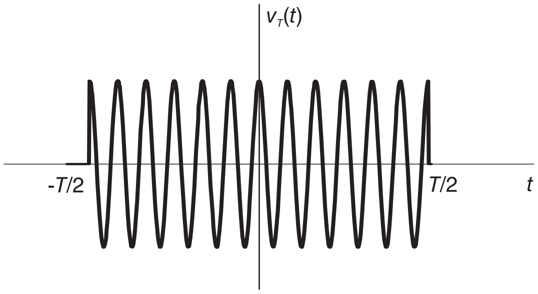


FIGURE 12.3
Truncated sinusoid.

We have

$$\Pi_{T/2}\left(t\right)\overset{\mathcal{F}}{\longleftrightarrow}T\ Sa\left(\omega T/2\right)$$

$$V_T\left(j\omega\right)\triangleq\mathcal{F}\left[v_T(t)\right]=\frac{AT}{2}\left\{Sa\left[\left(\omega-\omega_c\right)T/2\right]+Sa\left[\left(\omega+\omega_c\right)T/2\right]\right\}$$

wherefrom the energy spectral density is given by

$$\begin{aligned}\varepsilon_{v_Tv_T}\left(\omega\right)=\left|V_T\left(j\omega\right)\right|^2=\left(A^2T^2/4\right)\big\{&Sa^2\left[\left(\omega-\omega_c\right)T/2\right]\\&+Sa^2\left[\left(\omega+\omega_c\right)T/2\right]+2Sa\left[\left(\omega-\omega_c\right)T/2\right]Sa\left[\left(\omega+\omega_c\right)T/2\right]\big\}\end{aligned}$$

and is shown graphically in Fig. 12.4.

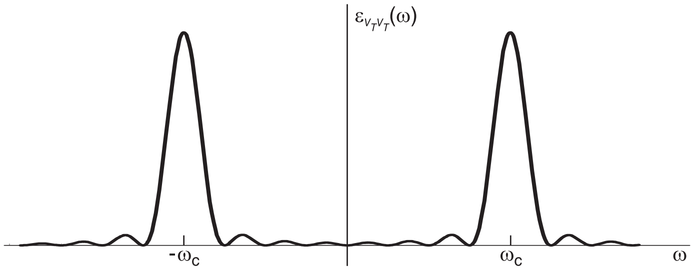


FIGURE 12.4
Energy spectral density.

12.2 Average, Energy and Power of Continuous-time Signals

The *average normalized power*, or simply *average power*, of a signal $f(t)$ is defined by

$$\overline{f^2}(t) \triangleq \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |f(t)|^2 dt. \tag{12.17}$$

The energy E, as seen above, is given by

$$E = \int_{-\infty}^{\infty} f^2(t)dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(j\omega)|^2 d\omega. \tag{12.18}$$

A signal which has a finite energy E has an average power $\overline{f^2}(t)$ of zero. Such a signal is called an *energy signal*.

A *power signal* is one that has infinite energy and finite non-nil average power, i.e. $0 < \overline{f^2}(t) < \infty$. A periodic signal is a power signal. Its average power P is evaluated as its power over one period.

Let $f(t)$ be periodic of period T_0. Its average normalized power, or simply average power, is given by

$$P = \overline{f^2}(t) = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} |f(t)|^2 \, dt = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} f(t) f^*(t) dt. \tag{12.19}$$

From Parseval's relation for periodic functions

$$\frac{1}{T_0} \int_{-T_0/2}^{T_0/2} |f(t)|^2 dt = \sum_{n=-\infty}^{\infty} |F_n|^2 . \tag{12.20}$$

The average power of a periodic signal is thus given by the sum

$$P = \overline{f^2}(t) = \sum_{n=-\infty}^{\infty} |F_n|^2 . \tag{12.21}$$

12.3 Discrete-Time signals

For discrete-time signals the energy and average power are similarly defined. If a sequence $f[n]$ has finite energy, defined as,

$$E = \sum_{n=-\infty}^{\infty} f^2[n] \tag{12.22}$$

it is called an *energy signal.*

If it has a finite average power, defined as

$$P = \lim_{M \to \infty} \frac{1}{2M} \sum_{n=-M}^{M} f^2[n] \tag{12.23}$$

it is called a power signal.

If the sequence is periodic with period M its average power over one period is

$$P = \frac{1}{M} \sum_{n=0}^{M-1} f^2 [n] \,. \tag{12.24}$$

An impulsive signal

$$f(t) = \sum_{n=-\infty}^{\infty} f_n \delta (t - nT) \tag{12.25}$$

such as the one shown in Fig. 12.5 and which can be an ideal sampling of a continuous-time signal, is considered to be an energy signal if its average power defined as

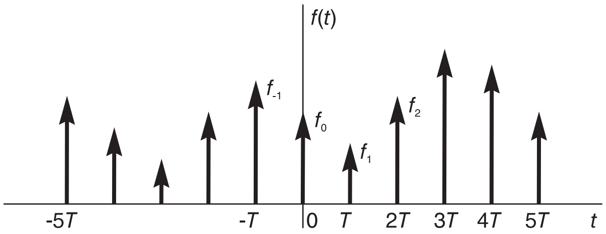


FIGURE 12.5
Impulsive signal.

$$\lim_{M \to \infty} \frac{1}{2MT} \sum_{n=-M}^{M} |f_n|^2 \tag{12.26}$$

is zero; otherwise it is a power signal.

12.4 Energy Signals

Let $f(t)$ and $g(t)$ be two real energy signals. We show that the Fourier transform of their cross-correlation function $r_{fg}(t)$ is equal to the cross spectral density $\varepsilon_{fg}(\omega)$.

We have already seen that correlation can be written as a convolution

$$r_{fg}(t) = \int_{-\infty}^{\infty} f(t + \tau) g(\tau) d\tau = f(t) * g(-t) \tag{12.27}$$

$$r_{fg}(-t) = r_{gf}(t) \,. \tag{12.28}$$

The Fourier transform of $r_{fg}(t)$ is therefore given by

$$R_{fg}(j\omega) = F(j\omega) G^*(j\omega) = \varepsilon_{fg}(\omega) \tag{12.29}$$

i.e. *the Fourier transform of the cross-correlation function of two energy signals is equal to their cross-energy spectral density.*

We note moreover, that if the functions $f(t)$ and $g(t)$ are complex then

$$r_{fg}(t) = \int_{-\infty}^{\infty} f(t+\tau)\, g^*(\tau)\, d\tau \tag{12.30}$$

$$R_{fg}(j\omega) \triangleq \mathcal{F}[r_{fg}(t)] = F(j\omega)\, G^*(j\omega) = \varepsilon_{fg}(\omega)\,. \tag{12.31}$$

Moreover

$$r_{fg}(-t) = r_{fg}^*(t)\,. \tag{12.32}$$

12.5 Auto-Correlation of Energy Signals

The Fourier transform of the autocorrelation function $r_{ff}(t)$ of an energy signal $f(t)$ is given by

$$R_{ff}(j\omega) = \mathcal{F}[r_{ff}(t)] = F(j\omega)\, F^*(j\omega) = |F(j\omega)|^2 = \epsilon_{ff}(\omega) \tag{12.33}$$

i.e.

$$r_{ff}(t) \xleftrightarrow{\mathcal{F}} |F(j\omega)|^2 = \varepsilon_{ff}(\omega) \tag{12.34}$$

$$\varepsilon_{ff}(\omega) = R_{ff}(j\omega) \tag{12.35}$$

so that the Fourier transform of the autocorrelation function of an energy signal is equal to the energy spectral density of the signal.

We note that the Fourier transform $F(j\omega)$ of a complex function $f(t)$ is not in general symmetric about origin, that is, $F(-j\omega) \neq F^*(j\omega)$. The energy spectral density $\varepsilon_{ff}(\omega) \triangleq |F(j\omega)|^2$ is real but not symmetric about the origin. Being real, however, its inverse transform is symmetric, that is, $r_{ff}(-t) = r_{ff}^*(t)$, as already established.

We note on the other hand that if the function $f(t)$ is real then $F(-j\omega) = F^*(j\omega)$ wherefrom the function $\varepsilon_{ff}(\omega) = |F(j\omega)|^2$ is even and its inverse transform $r_{ff}(t)$ is real (and even); $r_{ff}(-t) = r_{ff}(t)$.

Let $f(t)$ be generally complex. Writing

$$r_{ff,R}(t) \triangleq \Re[r_{ff}(t)]\,, \quad r_{ff,I}(t) \triangleq \Im[r_{ff}(t)] \tag{12.36}$$

$$r_{ff,R}(t) = r_{ff,R}(-t) \tag{12.37}$$

$$r_{ff,I}(t) = -r_{ff,I}(-t) \tag{12.38}$$

$$\begin{aligned} \varepsilon_{ff}(\omega) &= \int_{-\infty}^{\infty} r_{ff}(t) e^{-j\omega t} dt \\ &= \int_{-\infty}^{\infty} [r_{ff,R}(t) + j r_{ff,I}(t)]\, (\cos\omega t - j\sin\omega t)\, dt \\ &= 2\int_{0}^{\infty} (r_{ff,R}(t)\cos\omega t + r_{ff,I}(t)\sin\omega t) dt \end{aligned} \tag{12.39}$$

$$\begin{aligned} r_{ff}(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \varepsilon_{ff}(\omega)\, e^{j\omega t} d\omega \\ &= \frac{1}{2\pi}\left\{\int_{-\infty}^{\infty} \varepsilon_{ff}(\omega)\cos\omega t\ d\omega + j\int_{\infty}^{\infty} \varepsilon_{ff}(\omega)\sin\omega t\ d\omega\right\} \end{aligned} \tag{12.40}$$

i.e.

$$r_{ff,\,R}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \varepsilon_{ff}(\omega)\cos\omega t\ d\omega \tag{12.41}$$

$$r_{ff,I}\left(t\right)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\varepsilon_{ff}\left(\omega\right)\sin\omega t\;d\omega. \tag{12.42}$$

We note that

$$r_{ff}\left(0\right)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\varepsilon_{ff}\left(\omega\right)d\omega. \tag{12.43}$$

If the function $f\left(t\right)$ is real we have

$$r_{ff}\left(-t\right)=r_{ff}\left(t\right),\quad r_{ff,I}\left(t\right)=0 \tag{12.44}$$

$$\varepsilon_{ff}\left(\omega\right)=\left|F\left(j\omega\right)\right|^{2}=2\int_{0}^{\infty}r_{ff}\left(t\right)\cos\omega t\;dt \tag{12.45}$$

$$r_{ff}\left(t\right)=\frac{1}{\pi}\int_{0}^{\infty}\varepsilon_{ff}\left(\omega\right)\cos\omega t\;d\omega \tag{12.46}$$

and

$$r_{ff}\left(t\right)\le r_{ff}\left(0\right)=E \tag{12.47}$$

E being the energy of $f\left(t\right)$.

Example 12.3 *Show that* $R_{ff}\left(j\omega\right)=\varepsilon_{ff}\left(\omega\right)$ *for the rectangular window*

$$f\left(t\right)=\Pi_{T}\left(t\right)=u\left(t+T\right)-u\left(t-T\right).$$

The transform of $f\left(t\right)$ *is*

$$F\left(j\omega\right)=2T\;Sa\left(T\omega\right).$$

The spectral density is given by

$$\varepsilon_{ff}\left(\omega\right)=\left|F\left(j\omega\right)\right|^{2}=4T^{2}Sa^{2}\left(T\omega\right).$$

The autocorrelation of $f\left(t\right)$ *is the triangle*

$$r_{ff}\left(t\right)=\left(2T-\left|t\right|\right)\Pi_{2t}\left(t\right)\triangleq 2T\Lambda_{2T}\left(t\right)$$

where, we recall, $\Lambda_{x}\left(t\right)$ *is a centered triangle of height unity and total base width* $2x$*. Its Fourier transform is*

$$R_{ff}\left(j\omega\right)\triangleq\mathcal{F}\left[r_{ff}\left(t\right)\right]=\varepsilon_{ff}\left(\omega\right).$$

The spectral density and autocorrelation function are shown in Fig. 12.6.

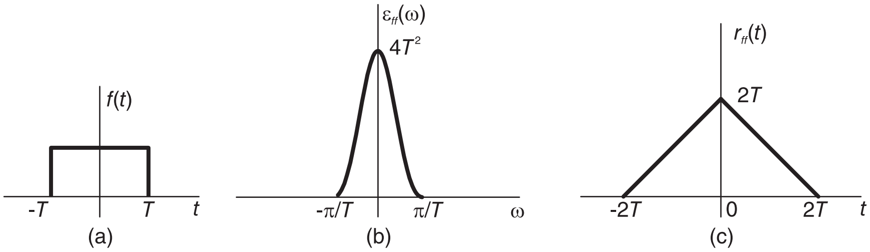


FIGURE 12.6
A rectangle, spectral density and autocorrelation function.

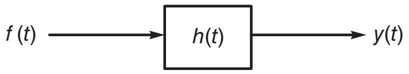


FIGURE 12.7
Linear system with input and output.

12.6 Energy Signal Through Linear System

Let an energy signal $f(t)$ be the input to a linear time invariant LTI system of impulse response $h(t)$, as shown in Fig. 12.7.

Let $r_{ff}(t)$ and $r_{yy}(t)$ be the autocorrelation of $f(t)$ and of $y(t)$, respectively. We have

$$R_{ff}(j\omega) = \mathcal{F}[r_{ff}(t)] = |F(j\omega)|^2 \tag{12.48}$$

$$R_{yy}(j\omega) = \mathcal{F}[r_{yy}(t)] = |Y(j\omega)|^2 . \tag{12.49}$$

Now

$$Y(j\omega) = F(j\omega)H(j\omega) \tag{12.50}$$

wherefrom

$$R_{yy}(j\omega) = |F(j\omega)|^2 |H(j\omega)|^2 \tag{12.51}$$

i.e.

$$R_{yy}(j\omega) = R_{ff}(j\omega)|H(j\omega)|^2 = R_{ff}(j\omega)H(j\omega)H^*(j\omega) . \tag{12.52}$$

Hence

$$\varepsilon_{yy}(\omega) = \varepsilon_{ff}(\omega)|H(j\omega)|^2 . \tag{12.53}$$

Moreover

$$\mathcal{F}^{-1}[H^*(j\omega)] = h(-t) \tag{12.54}$$

we have

$$r_{yy}(t) = r_{ff}(t) * h(t) * h(-t) \tag{12.55}$$

i.e. the autocorrelation of the system response is the convolution of the input signal autocorrelation with the convolution $h(t) * h(-t)$.

12.7 Impulsive and Discrete-Time Energy signals

Let $f_s(t)$ be a signal formed of equidistant impulses such as the signal

$$\begin{aligned} f_s(t) &= \ldots + f[-1]\delta(t+T) + f[0]\delta(t) + f[1]\delta(t-T) + \ldots && (12.56) \\ &= \sum_{n=-\infty}^{\infty} f[n]\delta(t-nT) && (12.57) \end{aligned}$$

shown in Fig. 12.8 (a).

We may view the impulsive signal $f_s(t)$ as the result of sampling a continuous-time signal $f_c(t)$ with a sampling interval of T seconds.

$$f_s(t) = f_c(t)\sum_{n=-\infty}^{\infty}\delta(t-nT) = \sum_{n=-\infty}^{\infty} f_c(nT)\delta(t-nT) . \tag{12.58}$$

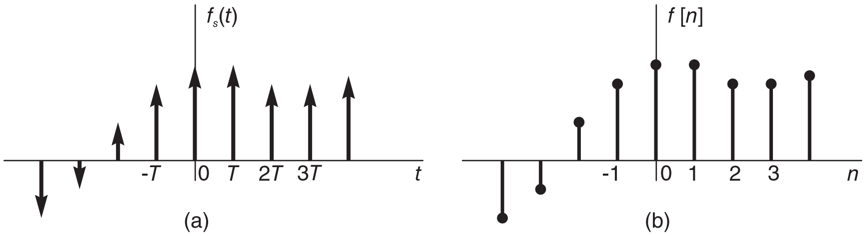


FIGURE 12.8
Signal with equidistant impulses and discrete-time signal counterpart.

Associated with $f_c(t)$ and $f_s(t)$ we also have a discrete-time function, namely, the sequence $f[n] = f_c(nT)$ shown in Fig. 12.8 (b). The energy of the signal $f_s(t)$ as well as that of $f[n]$ are defined by the summation

$$E = \sum_{n=-\infty}^{\infty} |f[n]|^2 . \tag{12.59}$$

If the energy is finite then the signal $f_s(t)$ and the sequence $f[n]$ are energy signals. The auto-correlation of the signal $f_s(t)$ can be obtained by evaluating the auto-correlation $r_{ff}[n]$ of the corresponding sequence $f[n]$. In fact the auto-correlation of $f_s(t)$ is given by

$$\begin{aligned} r_{f_s f_s}(t) &= \int_{-\infty}^{\infty} f_s(\tau) f_s(t+\tau)\, d\tau \\ &= \int_{-\infty}^{\infty} \sum_{m=-\infty}^{\infty} f[m]\, \delta(\tau - mT) \sum_{i=-\infty}^{\infty} f[i] \delta(t+\tau - iT)\, d\tau \\ &= \int_{-\infty}^{\infty} \sum_m \sum_i f[m] f[i]\, \delta(\tau - mT)\, \delta(t+\tau - iT)\, d\tau \\ &= \sum_m \sum_i f[m] f[i] \int_{\infty}^{\infty} \delta(\tau - mT) \delta(t+\tau - iT)\, d\tau \\ &= \sum_{m=-\infty}^{\infty} \sum_{i=-\infty}^{\infty} f[m] f[i]\, \delta(t - (i-m)T) . \end{aligned}$$

Letting $i - m = n$ we have

$$r_{f_s f_s}(t) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} f[m] f[m+n]\, \delta(t - nT) . \tag{12.60}$$

Interchanging the order of summations

$$r_{f_s f_s}(t) = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} f[m]\, f[m+n]\, \delta(t - nT) = \sum_{n=-\infty}^{\infty} \rho_n \delta(t - nT) \tag{12.61}$$

where

$$\rho_n = \sum_{m=-\infty}^{\infty} f[m] f[m+n] . \tag{12.62}$$

On the other hand the discrete auto-correlation of the corresponding sequence $f[n]$ is given by

$$r_{ff}[n] = \sum_{m=-\infty}^{\infty} f[m] f[n+m] . \tag{12.63}$$

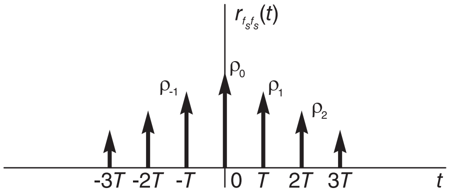


FIGURE 12.9
Auto-correlation of an impulsive signal.

Hence

$$\rho_n = r_{ff}[n]. \tag{12.64}$$

The autocorrelation $r_{f_sf_s}(t)$ is represented graphically in Fig. 12.9.

The auto-correlation of the impulsive signal $f_s(t)$ is therefore a one-to-one correspondence to the discrete auto-correlation of the corresponding discrete-time-sequence $f[n]$. It can be evaluated by simply effecting a discrete auto-correlation of the discrete sequence $f[n]$, followed by converting the resulting sequence $r_{ff}[n]$ into the corresponding impulsive function, which is the auto-correlation function $r_{f_sf_s}(t)$ of the function $f_s(t)$. The same approach can be used for evaluating the cross-correlation of two impulsive functions $f_s(t)$ and $g_s(t)$.

The Fourier transform of $f_s(t)$ is given by

$$F_s(j\omega) = \mathcal{F}\left[\sum_{n=-\infty}^{\infty} f[n]\,\delta(t-nT)\right] = \frac{1}{T}\sum_{n=-\infty}^{\infty} F_c\left(j\omega + j\frac{2\pi n}{T}\right). \tag{12.65}$$

This is equal to the Fourier transform $F\left(e^{j\Omega}\right)$ of the discrete-time counterpart, the sequence $f[n]$ with $\Omega = \omega T$.

$$F\left(e^{j\Omega}\right) = \sum_{n=-\infty}^{\infty} f[n]e^{-j\Omega n} = F_s(j\omega)\big|_{\omega=\Omega/T} = F_s\left(j\frac{\Omega}{T}\right). \tag{12.66}$$

The energy density $\varepsilon_{f_sf_s}(\omega)$ of the signal $f_s(t)$ is given by

$$\varepsilon_{f_sf_s}(\omega) = |F_s(j\omega)|^2 \tag{12.67}$$

and is therefore periodic of a period $2\pi/T$. Similarly the energy density of the sequence $f[n]$ is given by

$$\varepsilon_{ff}(\Omega) = \left|F\left(e^{j\Omega}\right)\right|^2 \tag{12.68}$$

and is periodic with a period 2π. The autocorrelation $r_{f_sf_s}(t)$ maybe written as the convolution:

$$r_{f_sf_s}(t) = f_s(t) \star f_s(t) = f_s(t) * f_s(-t) \tag{12.69}$$

$$R_{f_sf_s}(j\omega) = F_s(j\omega)\,F_s^*(j\omega) = |F_s(j\omega)|^2 = \varepsilon_{f_sf_s}(\omega) \tag{12.70}$$

$$r_{ff}[n] = f[n] \star f[n] = f[n] * f[-n] \tag{12.71}$$

$$R_{ff}\left(e^{i\Omega}\right) = F\left(e^{j\Omega}\right)F\left(e^{-j\Omega}\right) = \left|F\left(e^{j\Omega}\right)\right|^2 = \varepsilon_{ff}(\Omega). \tag{12.72}$$

The transform of the energy spectral density, is therefore given by

$$\varepsilon_{f_sf_s}(\omega) = R_{f_sf_s}(j\omega) = \mathcal{F}\left[\sum_{n=-\infty}^{\infty} \rho_n\delta(t-nt)\right] = \sum_{n=-\infty}^{\infty} \rho_n e^{-j\omega nT} \tag{12.73}$$

and

$$\varepsilon_{ff}(\Omega) = R_{ff}\left(e^{j\Omega}\right) = \sum_{n=-\infty}^{\infty} r_{ff}[n]e^{-j\Omega n}. \tag{12.74}$$

Since $f(t)$ is real we have $r_{ff}[-n] = r_{ff}[n]$ and $r_{f_s f_s}(-t) = r_{f_s f_s}(t)$, i.e., $\rho_{-n} = \rho_n$.

$$\varepsilon_{f_s f_s}(\omega) = \rho_0 + 2\sum_{n=1}^{\infty} \rho_n \cos nT\omega = r_{ff}[0] + 2\sum_{n-1}^{\infty} r_{ff}[n] \cos nT\omega \tag{12.75}$$

and

$$\varepsilon_{ff}(\Omega) = r_{ff}[0] + 2\sum_{n=1}^{\infty} r_{ff}[n] \cos n\,\Omega. \tag{12.76}$$

Example 12.4 *Let*

$$f_s(t) = \delta(t-T) + 2\delta(t-2T).$$

The signal is shown in Fig. 12.10 (a).

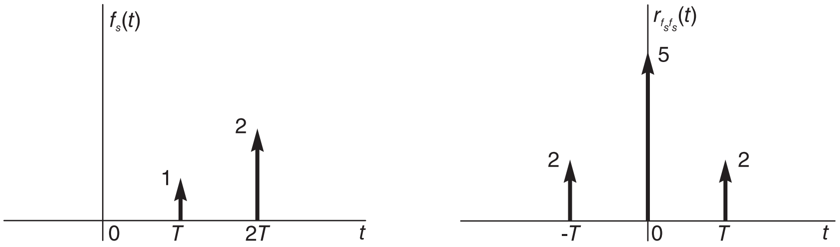
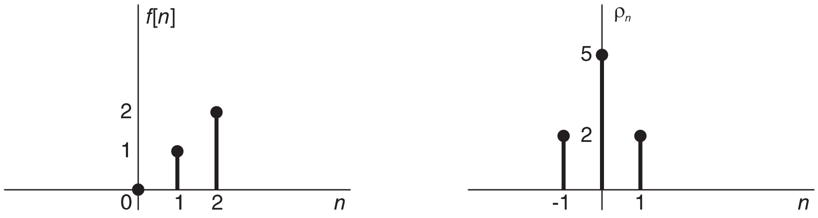


FIGURE 12.10
Impulsive signal and its autocorrelation.

Its autocorrelation is shown in Fig. 12.10 (b). The autocorrelation may be found by evaluating the auto correlation of the corresponding sequence $f[n] = \delta[n-1] + 2\delta[n-2]$. *We have*

$$\rho_n = r_{ff}[n] = \sum_{n=-\infty}^{\infty} f[n+m] f[m] = 2\delta[n+1] + 5\delta[n] + 2\delta[n-1].$$

The sequence $f[n]$ *and its auto correlation* $r_{ff}[n] = \rho_n$ *are shown in Fig. 12.11.*



(a) (b)

FIGURE 12.11
A sequence and its autocorrelation.

$$r_{f_s f_s}(t) = 5\delta(t) + 2\delta(t+T) + 2\delta(t-T)$$

$$\varepsilon_{f_s f_s}(\omega) = R_{f_s f_s}(j\omega) = 5 + 2e^{j\omega T} + 2e^{-j\omega T} = 5 + 4\cos T\omega.$$

Alternatively, we have

$$F_s(j\omega) = e^{-j\omega t} + 2e^{-j\omega t} = (\cos\omega T + 2\cos 2\omega T) - j(\sin\omega T + 2\sin 2\omega T)$$

$$\varepsilon_{f_s f_s}(\omega) = |F_s(j\omega)|^2 .$$

Similarly $\varepsilon_{ff}(\Omega) = R_{ff}\left(e^{j\Omega}\right) = 5 + 4\cos\Omega.$

Example 12.5 *Let*

$$f_c(t) = \begin{cases} t/10, & 0 \le t \le 30 \\ 6 - t/10, & 30 \le t \le 60. \end{cases}$$

Evaluate the sampled function $f_s(t)$*, the discrete-time function* $f[n]$ *and their auto-correlations, assuming a sampling interval of* $T = 10$ *sec. We have*

$$f_s(t) = \delta(t-T) + 2\delta(t-2T) + 3\delta(t-3T) + 2\delta(t-4T) + \delta(t-5T)$$

$$f[n] = f_c(nT) = f_c(10n) = \begin{cases} n, & 0 \le n \le 3 \\ 6 - n, & 3 \le n \le 6 \end{cases}$$

$$\begin{aligned}\rho_n = r_{ff}[n] &= \delta[n+4] + 4\,\delta[n+3] + 10\,\delta[n+2] + 16\,\delta[n+1] \\ &+ 19\,\delta[n] + 16\,\delta[n-1] + 10\,\delta[n-2] + 4\,\delta[n-3] + \delta[n-4].\end{aligned}$$

The sequence $f[n]$ *and its autocorrelation* $\rho[n] = r_{ff}[n]$ *are shown in Fig. 12.12 (a) and (b), respectively.*

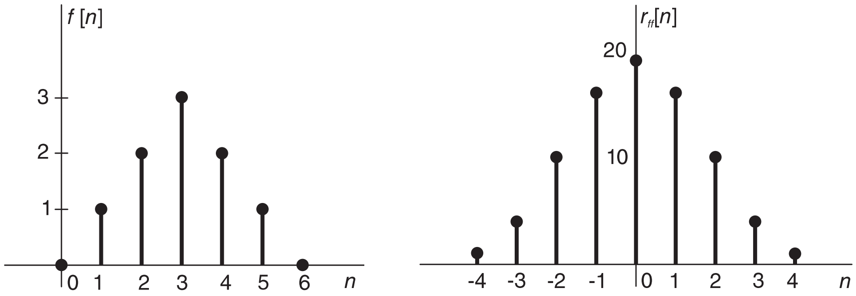


FIGURE 12.12
Sequence $f[n]$ and its autocorrelation.

The corresponding impulsive auto-correlation function $r_{f_s f_s}(t)$ *is deduced thereof to be*

$$\begin{aligned}r_{f_s f_s}(t) &= \delta(t+4T) + 4\delta(t+3T) + 10\delta(t+2T) + 16\delta(t+T) \\ &+ 19\delta(t) + 16\delta(t-T) + 10\delta(t-2T) + 4\delta(t-3T) + \delta(t-4T)\end{aligned}$$

$$\begin{aligned}\varepsilon_{f_s f_s}(\omega) &= R_{f_s f_s}(j\omega) \\ &= 19 + 32\cos T\omega + 20\cos 2T\omega + 8\cos 3T\omega + 2\cos 4T\omega \\ &= 19 + 32\cos 10\omega + 20\cos 20\omega + 8\cos 30\omega + 2\cos 40\omega\end{aligned}$$

$$\varepsilon_{ff}(\Omega) = R_{ff}\left(e^{i\Omega}\right) = 19 + 32\cos\Omega + 20\cos 2\Omega + 8\cos 3\Omega + 2\cos 4\Omega.$$

The energy spectral density $\varepsilon_{ff}(\Omega)$ *of the sequence* $f[n]$ *is shown in Fig. 12.13. Alternatively,*

$$F_s(j\omega) = e^{-j\omega T} + 2e^{-j2\omega T} + 3e^{-j3\omega T} + 2e^{-j4\omega T} + e^{-j5\omega T}$$

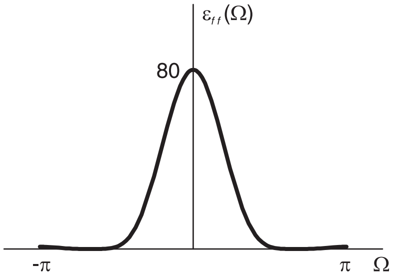


FIGURE 12.13
Energy spectral density.

$$\varepsilon_{f_s f_s}(\omega) = |F_s(j\omega)|^2 = F_s(j\omega) F_s^*(j\omega).$$

Letting

$$z = e^{j\omega T}, \quad z^* = e^{-j\omega T} = z^{-1}.$$

We have, with $z = e^{j\Omega}$,

$$\begin{aligned}\varepsilon_{f_s f_s}(\omega) &= \left(z^{-1} + 2z^{-2} + 3z^{-3} + 2z^{-4} + z^{-5}\right)\\ &\quad \left(z + 2z^2 + 3z^3 + 2z^4 + z^5\right)\\ &= 19 + 16z^{-1} + 10z^{-2} + 4z^{-3} + z^{-4} + 16z + 10z^2 + 4z^3 + z^4\\ &= 19 + 32\cos\omega T + 20\cos 2\omega T + 8\cos 3\omega T + 2\cos 4\omega T = R_{f_s f_s}(j\omega)\\ \varepsilon_{ff}(\Omega) &= 19 + 32\cos\Omega + 20\cos 2\Omega + 8\cos 3\Omega + 2\cos 4\Omega = R_{ff}\left(e^{j\Omega}\right).\end{aligned}$$

12.8 Powers Signals

We have seen that a power signal has a finite average power

$$0 < \overline{f^2}(t) < \infty, \tag{12.77}$$

where

$$\overline{f^2}(t) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |f(t)|^2 dt \tag{12.78}$$

and that a periodic signal is a power signal having an average power evaluated over one period

$$P = \overline{f^2}(t) = \frac{1}{T} \int_{-T/2}^{T/2} |f(t)|^2 dt = \sum_{n=-\infty}^{\infty} |F_n|^2. \tag{12.79}$$

In the following the cross and auto-correlation of such signals are defined.

12.9 Cross-Correlation

Let $f(t)$ and $g(t)$ be two real power signals. The cross-correlation $r_{fg}(t)$ is given by

$$r_{fg}(t) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f(t+\tau) g(\tau)\, d\tau \tag{12.80}$$

$$r_{fg}(-t) = r_{gf}(t) \tag{12.81}$$

as is the case for energy signals. If $f(t)$ and $g(t)$ are complex then

$$r_{fg}(t) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f(t+\tau)\, g^*(\tau)\, d\tau \tag{12.82}$$

$$r_{fg}(-t) = r^*_{gf}(t) \tag{12.83}$$

$$r_{ff}(t) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f(\tau) f(t+\tau)\, d\tau \tag{12.84}$$

$$r_{ff}(-t) = r_{ff}(t) \tag{12.85}$$

and

$$r_{ff}(0) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |f(t)|^2 dt = \overline{f^2}(t). \tag{12.86}$$

12.9.1 Power Spectral Density

For a real power signal $f(t)$ the power spectral density denoted by $S_{ff}(\omega)$ is by definition the Fourier transform of the autocorrelation function.

$$S_{ff}(\omega) = \mathcal{F}[r_{ff}(t)] = R_{ff}(j\omega). \tag{12.87}$$

Since $r_{ff}(t)$ is real and even its transform $S_{ff}(\omega)$ is real and even. We have

$$S_{ff}(\omega) = 2\int_0^\infty r_{ff}(t) \cos\omega t\ dt \tag{12.88}$$

and

$$r_{ff}(t) = \frac{1}{\pi} \int_0^\infty S_{ff}(\omega) \cos\omega t\ d\omega. \tag{12.89}$$

Let

$$f_T(t) = f(t)\, \Pi_T(t) = f(t)\{u(t+T) - u(t-T)\} \tag{12.90}$$

that is, $f_T(t)$ is a truncation of $f(t)$.

We have

$$F_T(j\omega) = \mathcal{F}[f_T(t)] = \int_{-T}^{T} f(t) e^{-j\omega t} dt. \tag{12.91}$$

The average power density over the interval $(-T,\ T)$ is the energy over the interval divided by the duration $2T$. Denoting it by $S_T(\omega)$ we have

$$S_T(\omega) \triangleq \frac{1}{2T} |F_T(j\omega)|^2. \tag{12.92}$$

It can be shown that $S_{ff}(\omega)$ is the limit as T tends to infinity of $S_T(\omega)$

$$S_{ff}(\omega) = \lim_{T\to\infty} S_T(\omega) = \lim_{T\to\infty} \frac{1}{2T} |F_T(j\omega)|^2. \tag{12.93}$$

In fact

$$\begin{aligned} S_{ff}(\omega) &= \mathcal{F}[r_{ff}(t)] = \mathcal{F}\left[\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f_T(t+\tau) f_T(\tau)\, d\tau\right] \\ &= \lim_{T\to\infty} \frac{1}{2T} \int_{-\infty}^{\infty} \int_{-T}^{T} f_T(t+\tau) f_T(\tau) d\tau\ e^{-j\omega t} dt \\ &= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f_T(\tau) \int_{-\infty}^{\infty} f_T(t+\tau)\, e^{-j\omega t} dt\ d\tau. \end{aligned} \tag{12.94}$$

Let

$$t + \tau = x \tag{12.95}$$

$$\begin{aligned}
S_{ff}(\omega) &= \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} f_T(\tau)\int_{-\infty}^{\infty} f_T(x)\, e^{-j\omega(x-\tau)}dx\ d\tau \\
&= \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} f_T(\tau)e^{j\omega\tau}F_T(j\omega)\, d\tau \\
&= \lim_{T\to\infty}\frac{1}{2T}F_T(j\omega)\int_{-T}^{T} f(\tau)e^{j\omega\tau}d\tau = \lim_{T\to\infty}\frac{1}{2T}F_T(j\omega)\, F_T^*(j\omega) \\
&= \lim_{T\to\infty}\frac{1}{2T}\left|F_T(j\omega)\right|^2 = \lim_{T\to\infty} S_T(\omega)\,. \qquad (12.96)
\end{aligned}$$

12.10 Power Spectrum Conversion of a Linear System

Let $f(t)$ be a power signal applied to the input of a linear time invariant LTI system the impulse response of which $h(t)$ is an energy signal. The system response may be written

$$y(t) = f(t) * h(t) = \int_{-\infty}^{\infty} f(\tau)h(t-\tau)\, d\tau. \qquad (12.97)$$

Let $r_{ff}(t)$ and $S_{ff}(\omega)$ be the autocorrelation and spectral density respectively of the input $f(t)$. The autocorrelation of the output signal $y(t)$ is given by

$$\begin{aligned}
r_{yy}(t) &= \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} y(\tau)y(t+\tau)\, d\tau \\
&= \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} h(u)f(\tau-u)\, du\int_{-\infty}^{\infty} h(x)f(t+\tau-x)\, dx\ dt.
\end{aligned} \qquad (12.98)$$

Interchanging the order of integration

$$\begin{aligned}
r_{yy}(t) &= \lim_{T\to\infty}\frac{1}{2T}\int_{-\infty}^{\infty} h(u)\int_{-\infty}^{\infty} h(x)\int_{-T}^{T} f(\tau-u)f(t+\tau-x)\, d\tau\ dx\ du \\
&= \lim_{T\to\infty}\frac{1}{2T}\int_{-\infty}^{\infty} h(u)\int_{-\infty}^{\infty} h(x)\int_{-T-u}^{T-u} f(\alpha)f(\alpha+u+t-x)\, d\tau\ dx\ du \\
&= \int_{-\infty}^{\infty} h(u)\int_{-\infty}^{\infty} h(x)r_{ff}(u+t-x)\, dx\ du. \qquad (12.99)
\end{aligned}$$

We note that the second integral is a convolution. Writing

$$z(u+t) = \int_{-\infty}^{\infty} h(x)r_{ff}(u+t-x)\, dx = h(t) * r_{ff}(u+t) \qquad (12.100)$$

i.e.

$$z(t) = h(t) * r_{ff}(t) \qquad (12.101)$$

we have

$$r_{yy}(t) = \int_{-\infty}^{\infty} h(u)z(u+t)\, du = r_{zh}(t) = z(t) * h(-t) = r_{ff}(t) * h(t) * h(-t)\,. \qquad (12.102)$$

We conclude that the system response $y(t)$ is a power signal the autocorrelation $r_{yy}(t)$ of which is the convolution of the input signal autocorrelation $r_{ff}(t)$ with the function $h(t) * h(-t)$ that is, the convolution of $h(t)$ with is reflection. Moreover,

$$S_{yy}(\omega) = \mathcal{F}[r_{yy}(t)] = \mathcal{F}[r_{ff}(t)] \cdot H(j\omega)\, H^*(j\omega) = S_{ff}(\omega)\left|H(j\omega)\right|^2. \qquad (12.103)$$

We conclude that the time domain convolution $y(t) = f(t) * h(t)$ leads to the power spectral density transformation

$$S_{yy}(\omega) = S_{ff}(\omega) |H(j\omega)|^2 \tag{12.104}$$

and that more generally, the convolution $y(t) = f(t) * x(t)$ of a power signal $f(t)$ and an energy signal $x(t)$ leads to the power spectral density transformation

$$S_{yy}(\omega) = S_{ff}(\omega) |X(j\omega)|^2 . \tag{12.105}$$

In the case of input white noise for example

$$S_{ff}(\omega) = 1 \tag{12.106}$$

wherefrom $r_{ff}(t) = \delta(t)$ and $S_{yy}(\omega) = |H(j\omega)|^2$, i.e. the power density of the system response is equal to the energy density of the impulse response $h(t)$.

Example 12.6 *Let $f(t) = K$, where K is a constant. The autocorrelation of $f(t)$ given by*

$$r_{ff}(t) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} K^2 dt = K^2$$

is a constant, and

$$S_{ff}(\omega) = \mathcal{F}[r_{ff}(t)] = R_z z j\omega = 2\pi K^2 \delta(\omega)$$

as shown in Fig. 12.14.

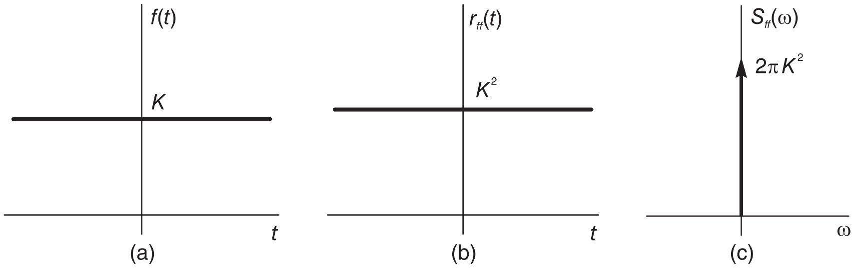


FIGURE 12.14
A constant, autocorrelation and power spectral density.

The power by direct evaluation is $P = K^2$ and, alternatively,

$$P = \overline{f^2}(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{ff}(\omega) d\omega = K^2.$$

Note that functions that are absolutely integrable such $e^{-t}u(t)$ have finite energy and thus represent energy signals whereas functions such as the step function and unity represent power signals.

Example 12.7 *Evaluate the autocorrelation and spectral density of the signal*

$$f(t) = Ku(t).$$

The signal is shown in Fig. 12.15(a).

$$r_{ff}(t) = \lim_{T\to\infty} \frac{K^2}{2T} \int_{-T}^{T} u(\tau) u(t+\tau) d\tau.$$

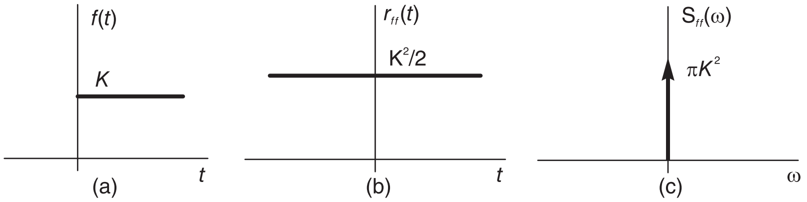


FIGURE 12.15
Unit step function, autocorrelation and power spectral density.

Consider the integral

$$I = \int_{-T}^{T} u(\tau)u(t+\tau)\,d\tau$$

and the case $t > 0$. *We have*

$$I = \int_{o}^{T} d\tau = T$$

and

$$r_{ff}(t) = \lim_{T\to\infty} \frac{K^2}{2T} I = \frac{K^2}{2}, \quad t > 0.$$

For $t < 0$ *we can use the symmetry property*

$$r_{ff}(-t) = r_{ff}(t) = K^2/2$$

wherefrom

$$r_{ff}(t) = K^2/2, \quad \forall t$$

and

$$S_{ff}(\omega) = R_{ff}(j\omega) = \pi K^2 \delta(\omega).$$

The autocorrelation and spectral density are shown in Fig. 12.15(b) and (c), respectively.

12.11 Impulsive and Discrete-Time Power Signals

Let $f(t)$ be the impulsive function

$$f_s(t) = \sum_{n=-\infty}^{\infty} f[n]\delta(t-nT). \tag{12.107}$$

If the average power of $f(t)$ is finite and not zero, that is,

$$0 < \lim_{N\to\infty} \frac{1}{2N} \sum_{n=-N}^{N-1} |f[n]|^2 < \infty \tag{12.108}$$

then $f(t)$ is a power signal. As noted earlier $f_s(t)$ may be the result of ideal sampling of a continuous-time function $f_c(t)$

$$f_s(t) = \sum_{n=-\infty}^{\infty} f_c(nT)\delta(t-nT). \tag{12.109}$$

The discrete-time representation of the same signal is the sequence $f[n]$ defined by $f[n] = f_c(nT)$. The autocorrelation of $f_s(t)$ is given by

$$r_{f_s f_s}(t) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f_s(\tau) f_s(t+\tau)\, d\tau. \tag{12.110}$$

As in the case of impulsive and discrete-time energy signals it can be shown that

$$r_{f_s f_s}(t) = \sum_{n=-\infty}^{\infty} \rho_n \delta(t - nT) \tag{12.111}$$

where

$$\rho_n = \lim_{M\to\infty} \frac{1}{2MT} \sum_{m=-M}^{M-1} f[m] f[m+n]. \tag{12.112}$$

The power density is given by

$$\begin{aligned} S_{f_s f_s}(\omega) &= \mathcal{F}\left[r_{f_s f_s}(t)\right] \triangleq R_{f_s f_s}(j\omega) = \mathcal{F}\left[\sum_{n=-\infty}^{\infty} \rho_n \delta(t - nT)\right] \\ &= \sum_{n=-\infty}^{\infty} \rho_n e^{-jnT\omega} = \rho_0 + 2\sum_{n=1}^{\infty} \rho_n \cos nT\omega. \end{aligned} \tag{12.113}$$

For the sequence $f[n]$ the autocorrelation is given by

$$r_{ff}[n] = \lim_{M\to\infty} \frac{1}{2M} \sum_{m=-M}^{M-1} f[m] f[n+m] \tag{12.114}$$

so that

$$\rho_n = \frac{1}{T} r_{ff}[n] \tag{12.115}$$

$$\begin{aligned} S_{ff}(\Omega) &= \mathcal{F}\left[r_{ff}[n]\right] = R_{ff}(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} r_{ff}[n] e^{-j\Omega n} \\ &= r_{ff}[0] + 2\sum_{n=1}^{\infty} r_{ff}[n] \cos \Omega\, n. \end{aligned} \tag{12.116}$$

12.12 Periodic Signals

Let a real signal $f(t)$ be periodic of period T. Its autocorrelation $r_{ff}(t)$ is periodic defined by

$$\begin{aligned} r_{ff}(t) &= \frac{1}{T}\int_0^T f(\tau) f(t+\tau)\, d\tau = \frac{1}{T}\int_0^T f(\tau) \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0(t+\tau)} d\tau \\ &= \frac{1}{T} \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 \tau} \int_0^T f(\tau) e^{jn\omega_0 \tau} d\tau = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t} F_n^* \end{aligned} \tag{12.117}$$

i.e.

$$r_{ff}(t) = \sum_{n=-\infty}^{\infty} |F_n|^2 e^{jn\omega_0 t}, \quad \omega_0 = 2\pi/T \tag{12.118}$$

which has the form of a Fourier series expansion having as coefficients $|F_n|^2$. We can therefore write

$$|F_n|^2 = \frac{1}{T} \int\limits_T r_{ff}(t) e^{-jn\omega_0 t} dt \tag{12.119}$$

$$r_{ff}(t) = \sum_{n=-\infty}^{\infty} |F_n|^2 \cos n\omega_0 \, t \tag{12.120}$$

$$r_{ff}(t) = |F_0|^2 + 2\sum_{n=1}^{\infty} |F_n|^2 \cos n \, \omega_0 t. \tag{12.121}$$

The power spectral density is given by

$$S_{ff}(\omega) = R_{ff}(j\omega) = 2\pi \sum_{n=-\infty}^{\infty} |F_n|^2 \delta(\omega - n\omega_0). \tag{12.122}$$

The average power of $f(t)$ is given by

$$P = \overline{f^2(t)} = r_{ff}(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} R_{ff}(j\omega) d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{ff}(\omega) d\omega. \tag{12.123}$$

Moreover,

$$P = \frac{1}{T} \int_T f^2(t) \, dt = \sum_{n=-\infty}^{\infty} |F_n|^2. \tag{12.124}$$

Example 12.8 *Evaluate the power, the spectral density and autocorrelation function of the signal* $f(t) = A\cos\omega_0 t$ *where* $\omega_0 = 2\pi/T$. *We have*

$$P = \frac{1}{T} \int_0^T A^2 \cos^2 \omega_0 t \; dt = \frac{A^2}{T} \times \frac{1}{2} \int_0^T (\cos 2\omega_0 t + 1) dt = A^2/2.$$

The evaluation of the average power of a sinusoid is often needed. It is worth while remembering that the average power of a sinusoid of amplitude A is simply $A^2/2$.

We also note that the Fourier series coefficients of the expansion

$$f(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t}$$

are given by

$$F_n = \begin{cases} A/2, & n = \pm 1 \\ 0, & \textit{otherwise} \end{cases}$$

wherefrom

$$P = \overline{f^2(t)} = \sum |F_n|^2 = 2 \times A^2/4 = A^2/2$$

$$S_{ff}(\omega) = 2\pi \sum |F_n|^2 \delta(\omega - n\omega_0) = \pi \frac{A^2}{2} \{\delta(\omega - \omega_0) + \delta(\omega + \omega_0)\}$$

$$P = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\pi A^2}{2} \{\delta(\omega - \omega_0) + \delta(\omega + \omega_0)\} d\omega = A^2/2$$

$$r_{ff}(t) = |F_0|^2 + 2\sum_{1}^{\infty} |F_n|^2 \cos n \, \omega_0 t = (A^2/2) \cos \omega_0 t.$$

We note, moreover, that

$$R_{ff}(j\omega) = \frac{\pi A^2}{2} \{\delta(\omega - \omega_0) + \delta(\omega + \omega_0)\} = S_{ff}(\omega).$$

12.12.1 Response of an LTI System to a Sinusoidal Input

let $x(t) = \sin(\beta t + \theta)$ be the input to an LTI system. We evaluate the power spectral density at the input and output of the system.
The power spectral density of the input is

$$S_{xx}(\omega) = 2\pi \sum_{n=-\infty}^{\infty} |X_n|^2 \delta(\omega - n\omega_0). \tag{12.125}$$

where $\omega_0 = \beta$. The power spectral density of the output is

$$S_{yy}(\omega) = 2\pi \sum_{n=-\infty}^{\infty} |Y_n|^2 \delta(\omega - n\omega_0) == 2\pi \sum_{n=-\infty}^{\infty} |X_n|^2 |H(jn\beta)|^2 \delta(\omega - n\beta). \tag{12.126}$$

The average power of the input $x(t)$ is

$$P = \overline{x^2(t)} = \sum_{n=-\infty}^{\infty} |X_n|^2 = A^2/2. \tag{12.127}$$

and that of the output is

$$P = \overline{y^2(t)} = \sum_{n=-\infty}^{\infty} |Y_n|^2 = (A^2/2)|H(jn\beta)|^2 \tag{12.128}$$

Example 12.9 *The signal $x(t) = A\sin(\beta t)$, with $A = 1$ and $\beta = \pi$, is applied to the input of an LTI system of impulse response $h(t) = \Pi_{0.5}(t)$. Is the system response $y(t)$ an energy or power signal? Evaluate the energy and power, and the spectral density at the system input and output.*
The input signal $x(t)$ and response $y(t)$ have infinite energy and are hence power signals. since their energy is infinite. The spectral densities are

$$S_x(\omega) = (\pi/2)[\delta(\omega - \pi) + \delta(\omega + \pi)]$$

and

$$S_y(\omega) = S_x(\omega)|H(j\omega)|^2 = \frac{\pi}{2} Sa^2(\pi/2)[\delta(\omega - \pi) + \delta(\omega + \pi)] = 0.637[\delta(\omega - \pi) + \delta(\omega + \pi)]$$

The input power is

$$P_x = \overline{x^2(t)} = \frac{1}{2\pi}\int_{\infty}^{\infty} S_x(\omega) d\omega = 0.5$$

The output power is

$$P_y = \overline{y^2(t)} = \frac{1}{2\pi}\int_{\infty}^{\infty} S_y(\omega) d\omega = 0.203$$

Alternatively, note that the input sinusoid Amplitude is $A = 1$ and its power is $P_x = \overline{x^2(t)} = A^2/2 = 0.5$. The output is $y(t) = A\underline{|H(j\pi)|}\sin(\beta t + arg[H(j\pi)]) = B\sin(\pi t + \theta)$, where $B = 0.6366$ and $\theta = -\pi/2$, and its power is $P_y = \overline{y^2(t)} = B^2/2 = 0.203$.

12.13 Power Spectral Density of an Impulse Train

Consider the impulse train shown in Fig. 12.16(a).

$$x(t) = \rho_T(t) \triangleq \sum_{n=-\infty}^{\infty} \delta(t - nT). \tag{12.129}$$

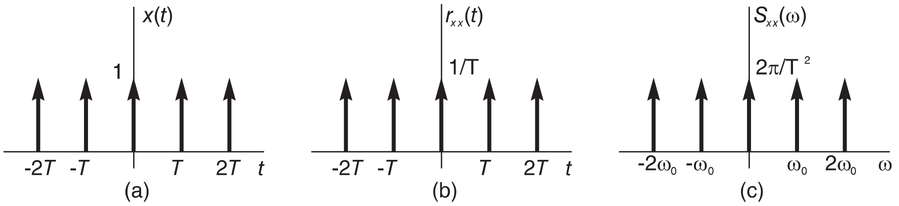


FIGURE 12.16
Impulse train, autocorrelation and power spectral density.

To evaluate the power spectral density of the impulse train we may proceed by applying the correlation definition directly over one period.

$$r_{xx}(t) = \frac{1}{T}\int_{-T/2}^{T/2} \delta(\tau)\delta(t+\tau)\,d\tau = \frac{1}{T}\delta(t), \quad -T/2 \le t \le T/2 \tag{12.130}$$

that is, $r_{xx}(t)$ is an impulse train of period T and impulses of intensity $1/T$

$$r_{xx}(t) = \frac{1}{T}\sum \delta(t-nT) = \frac{1}{T}\rho_T(t). \tag{12.131}$$

The power spectral density with $\omega_0 = 2\pi/T$ is given by

$$S_{xx}(\omega) = R_{xx}(j\omega) = \frac{1}{T}\omega_0\rho_{\omega_0}(\omega) = \frac{2\pi}{T^2}\sum_{n=-\infty}^{\infty}\delta(\omega - n\omega_0). \tag{12.132}$$

Alternatively, $X_n = 1/T$ and

$$S_{xx}(\omega) = 2\pi\sum_{n=-\infty}^{\infty}|X_n|^2\delta(\omega - n\omega_0) = \frac{2\pi}{T^2}\sum_{n=-\infty}^{\infty}\delta(\omega - n\omega_0). \tag{12.133}$$

Example 12.10 *Let $v(t)$ be the periodic ramp shown in Fig. 12.18. Evaluate the power spectral density. We have found in Chapter 2 that the fourier series coefficients are*

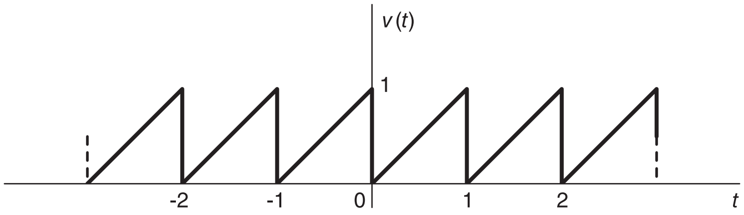


FIGURE 12.17
Periodic ramp.

$$V_n = \begin{cases} A/2, & n = 0 \\ jA/(2\pi n), & n \ne 0 \end{cases}$$

where $A = 1$ *and* $\omega_0 = 2\pi$. *Hence*

$$S_{vv}(\omega) = 2\pi\sum_{n=-\infty}^{\infty}|V_n|^2\,\delta(\omega - n\omega_0) = \left(\pi A^2/2\right)\delta(\omega) + \sum_{\substack{n=-\infty \\ n\ne 0}}^{\infty}\frac{A^2}{2\pi n^2}\delta(\omega - n\omega_0)$$

$$r_{vv}(t) = V_0^2 + 2\sum_{n=1}^{\infty} |V_n|^2 \cos n\omega_0 t = 1/4 + \sum_{n=1}^{\infty} \left(\frac{1}{2\pi^2 n^2}\right) \cos n\omega_0 t.$$

A direct evaluation of the periodic autocorrelation of the periodic ramp $v(t)$ by the usual shift-multiply-integrate process as shown in Fig. **??** *we obtain*

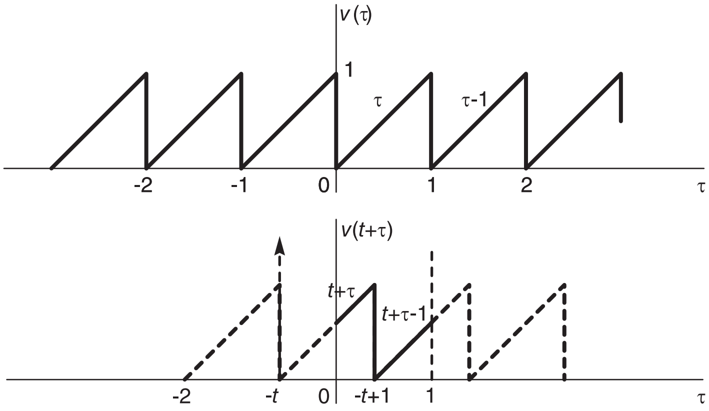


FIGURE 12.18
Periodic ramp and its shifting in time.

$$\begin{aligned} r_{vv}(t) &= \int_0^{1-t} (t+\tau)\,\tau \; d\tau + \int_{1-t}^{1} (t+\tau-1)\,\tau \; d\tau, \quad 0 < t < 1 \\ &= (1/6)\left(2 - 3t + 3t^2\right), \quad 0 < t < 1. \end{aligned}$$

A Fourier series expansion of $r_{vv}(t)$ as a verification produces the trigonometric coefficients

$$a_n = 2\int_0^1 (1/6)\left(2 - 3t + 3t^2\right) \cos n2\pi t \; dt = \frac{1}{2\pi^2 n^2}, \quad n \geq 1$$

and $a_0 = 1/2$ as expected. The functions $S_{vv}(\omega)$ and $r_{vv}(t)$ are shown in Fig. 12.19 and Fig. 12.20, respectively.

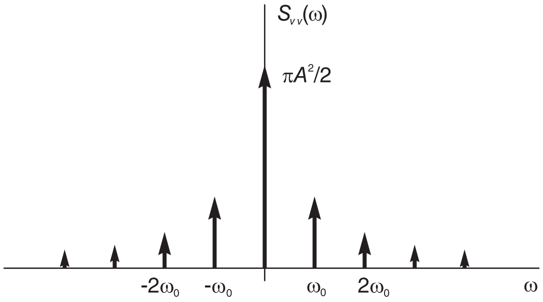


FIGURE 12.19
Power spectral density.

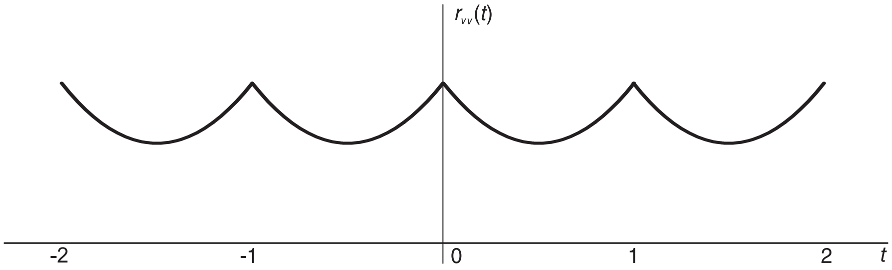


FIGURE 12.20
Autocorrelation of a periodic function.

Example 12.11 *Let*

$$v(t) = A\cos(m\omega_0 t + \theta), \quad m \text{ integer}$$

where $\omega_0 = 2\pi/T$. *Evaluate* $S_{vv}(\omega)$ *and* $r_{vv}(t)$.

We have

$$V_n = \begin{cases} (A/2)\,e^{j\theta}, & n = \pm m \\ 0, & \text{otherwise} \end{cases}$$

$$\begin{aligned} S_{vv}(\omega) &= 2\pi\left\{|V_m|^2\,\delta(\omega - m\omega_0) + |V_{-m}|^2\,\delta(\omega + m\omega_0)\right\} \\ &= \frac{\pi A^2}{2}\left\{\delta(\omega - m\omega_0) + \delta(\omega + m\omega_0)\right\} \end{aligned}$$

$$r_{vv}(t) = 2\left\{(A^2/4)\cos m\omega_0 t\right\} = (A^2/2)\cos m\omega_0 t.$$

12.14 Average, Energy and Power of a Sequence

As noted in Chapter 1 the average value of a sequence $x[n]$ is

$$\overline{x[n]} = \lim_{M\to\infty}\frac{1}{2M+1}\sum_{n=-M}^{M} x[n]. \tag{12.134}$$

A real sequence $x[n]$ is an energy sequence if it has a finite energy which can be defined as

$$E = \sum_{n=-\infty}^{\infty} x[n]^2. \tag{12.135}$$

A real aperiodic sequence $x[n]$ is a power sequence if it has a finite average power

$$P = \overline{x[n]^2} = \lim_{M\to\infty}\frac{1}{2M+1}\sum_{n=-M}^{M} x[n]^2. \tag{12.136}$$

If the sequence is periodic of period N its average power would be

$$P = \overline{x[n]^2} = \frac{1}{N}\sum_{n=0}^{N-1} x[n]^2. \tag{12.137}$$

Example 12.12 *Let the sequence* $x[n] = 3^{-n}\, u[n]$. *Evaluate its energy.*

$$E = \sum_{n=0}^{\infty} 3^{-2n} u[n] = \sum_{n=0}^{\infty} 9^{-n} u[n] = \frac{1}{1-9^{-1}} = \frac{9}{8}.$$

Example 12.13 *Evaluate the power of the signal*

$$x[n] = 10\cos(\pi n/8).$$

The period N *is deduced from*

$$x[n+N] = x[n]$$

$$10\cos(\pi n/8) = 10\cos[\pi(n+N)/8] = 10\cos(\pi n/8 + \pi N/8)$$

N *is the least value satisfying*

$$(\pi/8)\,N = 2\pi,\ 4\pi,\ 6\pi,\ \ldots$$

$$N = 16$$

$$\bar{P} = \frac{1}{16}\left\{100\sum_{n=0}^{15}\cos^2(\pi n/8)\right\} = \frac{100}{16}\times 2\sum_{n=0}^{7}\cos^2(\pi n/8)$$

$$= \frac{25}{2}(1 + 0.8536 + 0.5 + 0.1464 + 0 + 0.1464 + 0.5 + 0.8536) = 50.$$

12.15 Energy Spectral Density of a Sequence

The energy of a sequence $x[n]$is given by

$$E = \sum_{n=-\infty}^{\infty} |x[n]|^2.$$

The energy spectral density is given by $\varepsilon_x(\Omega) = \left|X\left(e^{j\Omega}\right)\right|^2$.

Pareseval's Relation states that

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left|X\left(e^{j\Omega}\right)\right|^2 d\Omega = \frac{1}{2\pi}\int_{-\infty}^{\infty}\varepsilon(\Omega)d\Omega.$$

12.16 Autocorrelation of an Energy Sequence

The autocorrelation of a real energy sequence is given by

$$r_{xx}[n] = \sum_{n=-\infty}^{\infty} x[n+m]\,x[m] = x[n] * x[-n].$$

Its Fourier transform is

$$R_{xx}\left(e^{j\Omega}\right) = X\left(e^{j\Omega}\right)X^*\left(e^{j\Omega}\right) = \left|X\left(e^{j\Omega}\right)\right|^2.$$

12.17 Power density of a Sequence

The power of a sequence is given by

$$P=\overline{x^2[n]}=\lim_{N\to\infty}\frac{1}{2N+1}\sum_{n=-N}^{N}|x[n]|^2.$$

The autocorrelation of a power sequence $x[n]$ is given by

$$r_{xx}[n]=\lim_{N\to\infty}\frac{1}{2N+1}\sum_{k=-N}^{N}x[n+k]x[k]$$

The power spectral density is given by

$$S_x(\Omega)=\mathcal{F}[r_{xx}[n]]=R_{xx}\left(e^{j\Omega}\right)$$

Parseval's relation takes the form

$$P=\overline{x^2[n]}=\lim_{N\to\infty}\frac{1}{2N+1}\sum_{n=-N}^{N}|x[n]|^2=\frac{1}{2\pi}\int_{-\infty}^{\infty}S_x(\Omega)d\Omega$$

12.18 Passage through a Linear System

Let $x[n]$ be the input and $y[n]$ the output of a linear time-invariant discrete-time system.

If $x[n]$ is an energy sequence its energy spectral density is $\varepsilon_x(\Omega)=\left|X\left(e^{j\Omega}\right)\right|^2$ and that of the output is

$$\varepsilon_y(\Omega)=\left|Y\left(e^{j\Omega}\right)\right|^2=\left|X\left(e^{j\Omega}\right)\right|^2\left|H\left(e^{j\Omega}\right)\right|^2.$$

If $x[n]$ is a [power sequence its energy spectral density is $S_x(\Omega)$ and that of the output is

$$S_y(\Omega)=S_x(\Omega)\left|H\left(e^{j\Omega}\right)\right|^2.$$

12.19 Problems

Problem 12.1 *A system has the impulse response*

$$h(t)=\sin\pi t\Pi_T(t)=\sin\pi t\left\{u(t)-u(t-T)\right\}.$$

The system receives the ideal impulse Train $\rho_T(t)$ as input

$$x(t)=\rho_T(t)=\sum_{n=-\infty}^{\infty}\delta(t-nT).$$

a) Evaluate the output $y(t)$ of the system if

i) $T=11$ sec,

ii) $T=12$ sec.

Evaluate its Fourier Transform $Y(j\omega)$ and its Fourier series expansion with analysis interval T.

b) With $T=12$ sec evaluate the energy and power spectral densities of $h(t)$ and $y(t)$. Write the expressions describing the auto-correlation of $h(t)$ and $y(t)$ in terms of their spectral densities.

Problem 12.2 *A signal $f(t)$ has a Fourier Transform*

$$F(j\omega) = 14\pi\delta(\omega) + j6\pi\delta\left(\omega - 2\pi \times 10^3\right) - j6\pi\delta\left(\omega + 2\pi \times 10^3\right) + 2\pi\delta\left(\omega - 8\pi \times 10^3\right) + 2\pi\delta\left(\omega + 8\pi \times 10^3\right).$$

a) Is the signal $f(t)$ an energy or power signal?
b) Evaluate the spectral density of $f(t)$.
c) What is the average power of $f(t)$?
d) What is the energy of the signal over an interval of 10^{-3} sec?
e) The signal $f(t)$ is filtered by an ideal bandpass filter with a pass-band $1000\pi < |\omega| < 6000\pi$ r/s and gain K. Evaluate the filter output $g(t)$. What is the average power of $g(t)$?

Problem 12.3 *Let*

$$x(t) = f(t) + g(t)$$

where

$$f(t) = A_1 \sin(\omega_1 t + \theta_1)$$
$$g(t) = A_2 \sin(\omega_2 t + \theta_2)$$

where $\omega_2 > \omega_1$.
a) Evaluate $S_x(\omega)$ the power spectral density of $x(t)$.
b) What is the average power of the component of $x(t)$ of frequency ω_2? A signal $y(t)$ is generated as

$$y(t) = f(t)\, g(t).$$

c) Evaluate the power spectral density $S_y(\omega)$.
d) The signal $y(t)$ is fed to a filter of frequency response

$$H(j\omega) = K\ \Pi_{\omega_2}(\omega).$$

Evaluate the power spectral density at the filter output $z(t)$.

Problem 12.4 *a) Evaluate the function $f(t)$ that is the inverse Laplace transform of the function*

$$F(s) = \left\{1 - e^{-(s+1)}\right\} / (s+1).$$

b) Evaluate the autocorrelation $r_{ff}(t)$ of the function $f(t)$ and its Fourier transform $R_{ff}(j\omega)$.
c) Can the Fourier transform $F(j\omega)$ of $f(t)$ be evaluated from $F(s)$ by letting $s = j\omega$? Justify your answer.
d) Evaluate $|F(j\omega)|^2$ and compare it with $R_{ff}(j\omega)$. e) Is $f(t)$ a power or energy signal ? Evaluate the energy or power spectral density of $f(t)$. Evaluate the energy / power of $f(t)$.
f) Let $H(s) = F(s)$ be the transfer function of a linear system. Let the input to the system be the signal

$$x(t) = \sum_{n=-\infty}^{\infty} \delta(t-n).$$

Evaluate the power spectral density of the system response $y(t)$. Evaluate the average power of $y(t)$ in the frequency band $0 < f < 1.5$ Hz.

Problem 12.5 *Consider a signal $x(t)$ of which the auto correlation function is given by*

$$r_{xx}(t) = e^{-|t|}, \quad -\infty < t < \infty.$$

a) Evaluate $\varepsilon_{xx}(\omega)$ the energy spectral density of $x(t)$.
b) Evaluate the total energy of $x(t)$.
c) The signal $x(t)$ is fed as the input of a filter of frequency response

$$H(j\omega) = \begin{cases} A, & 2 < |\omega| < 4 \\ 0, & \text{otherwise.} \end{cases}$$

Evaluate the total energy of the signal $y(t)$ at the filter output.

Problem 12.6

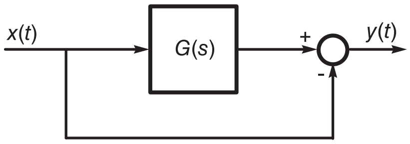


FIGURE 12.21
System block diagram.

In the system shown in Fig. 12.21 the transfer function $G(s)$ is that of a causal system and is given by

$$G(s) = 100\pi/(s + 100\pi).$$

a) Evaluate the system impulse response between the input $x(t)$ and the output $y(t)$
b) Given that the input is

$$x(t) = 1 + \cos 120\pi t$$

evaluate the average normalized power of the output $y(t)$. Evaluate the power spectral density of $y(t)$.

Problem 12.7 *Consider the signals*

$$x(t) = \sum_{n=-\infty}^{\infty} \{u(t - 2n) - u(t - 1 - 2n)\}$$

$$y(t) = e^{-t} u(t)$$

which represent voltage potentials in Volt as functions of time t in seconds.
a) For each of the two signals evaluate the total normalized energy and the average normalized power.
b) The signals $z(t)$ and $v(t)$ are given by $z(t) = x(t)\, y(t)$ and $v(t) = x(t) * y(t)$. For each of these signals state whether the signal is an energy or power signal, explaining why.

Problem 12.8 *The frequency transformation*

$$s \rightarrow (s^2 + 1)/s$$

is applied to a second order lowpass Butterworth filter prototype.
a) Write down the transfer functions $H_{LP}(s)$ and $H_{BP}(s)$ of the lowpass and bandpass filters.
b) Evaluate the central frequency ω_0 and the low and high edge frequencies ω_L and ω_H of the bandpass filter.
c) Re-write the values of $H_{LP}(s)$ and $H_{BP}(s)$ so that the filter maximal gain be 14 dB. Let the input to this bandpass filter be $x(t) = 10 + 7 \sin \omega_0 t$. Evaluate the average normalized power of the output $y(t)$.

Problem 12.9 *For each of the following signals, which are expressed in Volt as function of time in seconds, state whether it is an energy or power signal and evaluate its total normalized energy or average normalized power.*
a)
$v(t) = 3 \sin[1000\pi (t + 0.0025)] + 2 \cos(1500\pi t + \pi/5)$.
b) $w(t) = \begin{cases} 0.25(t-2), & 2 < t < 6 \\ 0, & \textit{otherwise.} \end{cases}$
c)
$$x(t) = \sum_{n=0}^{10} w(t - 10n).$$
d)
$$y(t) = \sum_{n=-\infty}^{\infty} w(t - 5n).$$

Problem 12.10 *Let $x(t)$ be a function, $X(j\omega)$ its Fourier transform and*

$$|X(j\omega)| = 1/\sqrt{1+\omega^2} + \pi/2\left\{\delta(\omega-\beta)+\delta(\omega+\beta)\right\}.$$

a) What is the average value of $x(t)$?
b) Is $x(t)$ periodic? If yes what is its period? If not why?
c) The signal $x(t)$ is applied as the input to a filter of frequency response $H(j\omega)$, where

$$|H(j\omega)| = \Pi_{2\beta}(\omega), \quad \arg\left[H(j\omega)\right] = -\pi\omega/(4\beta).$$

Sketch the amplitude spectrum $|Y(j\omega)|$ of the filter output $y(t)$.
d) Let $z(t) = x(t) + 0.5\sin(2.5\beta t) + 0.5$. Sketch the amplitude spectrum $|Z(j\omega)|$ of the signal $z(t)$.

Problem 12.11 *For each of the following signals evaluate the signal total energy and the average normalized power and deduce whether it is an energy or power signal:*
a) $v(t) = A\sin(2000\pi + \pi/3)$.
b) $w(t) = A\sin(2000\pi + \pi/3)R_{0.001}(t)$, where

$$R_{0.001}(t) = u(t) - u(t-0.001).$$

c) $x(t) = \sum_{n=-\infty}^{\infty} e^{-(t-5n)}\left\{u(t-5n) - u(t-5-5n)\right\}.$
d) $z(t) = A$.

Problem 12.12 *A system of transfer function*

$$H(s) = \left.\frac{K}{s+1}\right|_{s\,\rightarrow\, s/\omega_c}$$

receives an input $x(t)$ and produces an output $y(t)$. Assuming $x(t) = A\cos\omega_0 t$, where $A = 5$ Volt and $\omega_0 = 2\pi f_0 = 2\pi \times 500$ Hz.
a) With $K = 1$ and $\omega_c = 500\pi$ r/s, evaluate the average power of the signal $y(t)$.
b) With $K = 1$ find the value of ω_c so that the average power of $y(t)$ be 5 Watt.
c) With $\omega_c = 1000\pi$ r/s evaluate K so that the average power of $y(t)$ be 5 Watt.

Problem 12.13 *Given the signals $v(t) = x(t)\,y(t)$ and $f(t) = x(t) * z(t)$, where*

$$x(t) = 5R_3(t) = 5\left[u(t) - u(t-3)\right]$$

$$y(t) = 2\Pi_{0.5}(t) = 2\left[u(t+0.5) - u(t-0.5)\right]$$

$$z(t) = 1 + \cos(\pi t + \pi/3).$$

a) Evaluate $V(j\omega)$ and $F(j\omega)$, the Fourier transforms of $v(t)$ and $f(t)$ as well as the Fourier series coefficients F_n of $f(t)$.
b) State whether each of the signals $v(t)$ and $f(t)$ is an energy or power signal, evaluating the energy or power spectral density, the total energy or the average normalized power in each case.

Problem 12.14 *A signal $f(t)$ of average value $\overline{f(t)} = 15$ is applied to the input of a linear system of impulse response*

$$h(t) = 5e^{-7t}\sin 5\pi t\; u(t).$$

What is the average value $\overline{y(t)}$ of the system output $y(t)$?

Problem 12.15 *A signal $x(t)$ has a Fourier transform*

$$X(j\omega) = 2\pi\; Sa(\omega/400)\, e^{-j\omega/100} \sum_{n=-\infty}^{\infty} \delta(\omega - 100\pi n).$$

The signal is applied to the input of a filter of frequency response $H(j\omega)$ and output $y(t)$, where

$$|H(j\omega)| = \begin{cases} 1 - \left[(\omega - 300\pi) / (200\pi)\right]^2, & 100\pi < |\omega| < 500\pi \\ 0, & \text{otherwise} \end{cases}$$

$$\arg\left[H(j\omega)\right] = \begin{cases} -\pi/2, & \omega > 0 \\ \pi/2, & \omega < 0. \end{cases}$$

a) Evaluate the exponential Fourier series coefficients X_n of $x(t)$ with an analysis interval of 0.02 *sec.*
b) Sketch the frequency response $|H(j\omega)|$.
c) Evaluate the Fourier series coefficients Y_n of the output $y(t)$ over the same analysis period.
d) Evaluate the output $y(t)$ and the normalized average power of each components of $y(t)$.

Problem 12.16 *A system receives an input $x(t)$ and produces an output $y(t)$ that is the sum of $x(t)$ and a delayed version $x(t-\tau)$ where $\tau = 0.4 \times 10^{-3}$ sec. The signal $x(t)$ is a sinusoid of amplitude* 5 *Volt and frequency* 1 *kHz.*
a) Draw the block diagram describing the system.
b) Evaluate the impulse response $h(t)$ and frequency response $H(j\omega)$ of the system between its input $x(t)$ and output $y(t)$.
c) Evaluate and sketch the power spectral density $S_x(\omega)$ of the signal $x(t)$, expressed in terms of the Fourier series coefficients X_n of $x(t)$.
d) Evaluate and sketch the power spectral density $S_y(\omega)$ and the average power $\overline{y_2(t)}$ of the output $y(t)$.

Problem 12.17 *The signal $x(t) = e^{-7t}u(t)$ is applied to the input of a filter of frequency response $H(j\omega)$ given by*

$$H(j\omega) = \begin{cases} 5, & 1.1 \leqslant |\omega| \leqslant 3 \\ 0, & \text{otherwise} \end{cases}$$

Evaluate the energy spectral density $\varepsilon_x(\omega)$ of $x(t)$ and $\varepsilon_y(\omega)$ of $y(t)$.

Problem 12.18 *A filter of frequency response*

$$H(j\omega) = \left(1 - \omega^2/W^2\right)\Pi_W(\omega)$$

receives an input $v(t)$ and produces an output $y(t)$.

Assuming that the input $v(t)$ has an autocorrelation $r_{vv}(t) = \cos(Wt/4)$ evaluate the power spectral densities $S_{vv}(\omega)$ and $S_{yy}(\omega)$ of the signals $v(t)$ and $y(t)$, respectively. Evaluate the normalized average power of $y(t)$.

Problem 12.19 *Consider the signal*

$$v(t) = 10 \sin \beta t \Pi_{T/2}(t)$$

where $\beta = 4\pi/T$.
a) Sketch the signal $v(t)$. Evaluate its energy and normalized average power and corresponding spectral density if any.
b) What is the result of integrating the evaluated spectral density?

Problem 12.20 *A signal is given by*

$$v(t) = 10\cos\left[\beta(t-1)\right] + 5\sin\left[4\beta(t-2)\right] + 8\cos[10\beta(t-3)]$$

where $\beta = 2\pi/T$ and $T = 1$ sec.
a) Evaluate the exponential Fourier series coefficients of $v(t)$ with an analysis interval of one second.
b) Evaluate the signal power spectrum.

TABLE 12.1

Amplitude and phase spectra.

Frequency kHz	0	10	20	30	40	≥ 50
$\lvert F_n \rvert$Volt	2	2.5	3.5	2	1	0
$\arg[F_n]$ deg.	0	-10	-20	-30	-40	–

Problem 12.21 *A spectrum analyzer displays the amplitude spectrum in Volt and phase spectrum in degrees as the Fourier series coefficients* F_n *versus the frequency in Hz of a function* $f(t)$ *as shown in Table 12.1 and with* $F_{-n} = F_n^*$.

a) What is the period τ *and the average value of the function* $f(t)$*?*

b) Write the value of the function $f(t)$ *as a sum of real expressions.*

c) The signal $f(t)$ *is fed to a filter of frequency response* $H(j\omega)$ *where*

$$|H(j\omega)| = \Pi_B(\omega)$$

where $B = 50000\pi$ *rad/sec,* $\arg[H(j\omega)] = -(10^{-3}/180)\omega$ *rad/sec and the filter output* $g(t)$ *is modulated by the carrier* $\cos(40000\pi t)$ *producing an output* $y(t)$*. Sketch the Fourier transforms* $G(j\omega)$ *and* $Y(j\omega)$ *of* $g(t)$ *and* $y(t)$*.*

d) What is the average power of the output signal $y(t)$*?*

Problem 12.22 *Consider the signal:*

$$v(t) = u(t + t_0) - u(t - b + t_0)$$

where $b > t_0 > 0$.

a) Evaluate the autocorrelation $r_{vv}(t)$ *of* $v(t)$*.*

b) Evaluate the Fourier transform $R_{vv}(j\omega)$ *of* $r_{vv}(t)$*.*

c) Evaluate the Fourier transform $V(j\omega)$*, the energy spectral density and deduce therefrom the total energy of* $v(t)$*. Compare the result with* $R_{vv}(j\omega)$*.*

Problem 12.23 *Evaluate the energy spectral density for each of the following signals:*

a) $x(t) = e^t [u(t) - u(t-1)]$.

b) $y(t) = e^{-t} \sin(t) u(t)$.

Problem 12.24 *Given the signal* $v(t) = e^{-t} u(t)$

a) evaluate the energy of the signal $v(t)$*,*

b) evaluate the energy of the signal contained in the frequency range 0 *to* 1 *Hz.*

Problem 12.25 *Given the signal* $v(t) = e^{-t} u(t)$.

a) Show that $v(t)$ *is an energy signal.*

b) Evaluate the energy spectral density of $v(t)$*.*

c) Evaluate the normalized energy contained in the frequency range 0 *to* 1 *r/s.*

d) Evaluate the normalized energy contained in the frequency range 0 *to* 1 *Hz.*

e) Evaluate the auto-correlation function $r_{vv}(t)$ *of* $v(t)$*.*

f) Show how from $r_{vv}(t)$ *you can deduce the energy spectral density of* $v(t)$*.*

Problem 12.26 *The signal* $v(t) = 4e^{-2t} u(t)$ *is applied to the input of a filter of frequency response* $H(j\omega)$*.*

a) What is the total normalized energy E_v *of* $v(t)$*?*

b) What is the total normalized energy E_y *of the signal* $y(t)$ *at the filter output in the case where the filter is an ideal lowpass filter of unit gain and cut-off frequency* 2 *r/s?*

c) What is the total normalized energy E_y *of the signal* $y(t)$ *at the filter output in the case where the filter is an ideal bandpass filter of unit gain and pass band extending from* 1 *to* 2 *Hz?*

d) What is the total normalized energy E_y *of the signal* $y(t)$ *at the filter output in the case where the filter transfer function is* $H(s) = 1/(s+2)$*?*

e) What is the total normalized energy E_y *of the signal* $y(t)$ *at the filter output in the case where the filter frequency response is* $H(j\omega) = e^{-j\omega T}$*, where* T *is a constant?*

Problem 12.27 *Each of the following signals is given in Volt as a function of the time t in seconds. For each signal evaluate the total energy if it is an energy signal or the average power if it is a power signal.*

a) $x_a(t) = 3[u(t - T_a) - u(t - 6T_a)]$, *where* $T_a > 0$.

b) $x_b(t) = x_a(t)\cos(2\pi t/T_b)$, *where* $T_b = T_a$.

c) $x_c(t) = \sum_{n=-\infty}^{+\infty} x_b(t - nT_c)$, *where* $T_c = 15T_a$.

d) $x_d(t) = x_a(t) + 1$.

Problem 12.28 *Consider the three signals* $x(t)$, $y(t)$ *and* $z(t)$*:*

$$x(t) = u(t) - u(t-1)\,, \quad y(t) = u(t+0.5) - u(t-0.5)\,, \quad z(t) = \sin(\pi t)\,.$$

a) Is the sum $v(t) = x(t) + y(t)$ *an energy or power signal? Depending on the signal type, evaluate the total normalised energy or the average normalized power, respectively.*

b) Is the convolution $s(t) = x(t) * z(t)$ *an energy or power signal? Depending on the signal type, evaluate the energy spectral density or the power spectral density, respectively.*

Problem 12.29 *Evaluate the power spectral density and the average power of the following periodic signals:*

a) $v(t) = 5\cos(2000\pi t) + 3\sin(500\pi t)\,.$

b) $x(t) = [1 + \sin(100\pi t)]\cos(2000\pi t)\,.$

c) $y(t) = 4\sin^2(200\pi t)\cos(2000\pi t)\,.$

d) $z(t) = \sum_{n=-\infty}^{+\infty} 10^4\left(t - 10^{-3}n\right)\left\{u\left(t - 10^{-3}n\right) - u\left(t - 10^{-3}[n+1]\right)\right\}.$

Problem 12.30 *Let* $x(t)$ *be a periodic signal having a period* 5×10^{-2} *seconds. Its exponential Fourier series expansion with an analysis interval equal to its period has the Fourier series coefficient*

$$X_n = \begin{cases} 1, & n = 0,\ \pm 4 \\ \pm j, & n = \pm 1 \\ 0, & \text{otherwise.} \end{cases}$$

Let $y(t)$*, be a signal having the Fourier transform* $Y(j\omega) = 150/(125 + j\omega)$.

a) Let $z(t)$ *be the convolution* $z(t) = x(t) * y(t)$*. Evaluate the average power* $\overline{z^2(t)}$ *of* $z(t)$.

b) Let $v(t) = x(t) + y(t)$*. Evaluate the average power* $\overline{v^2(t)}$ *of* $v(t)$.

Problem 12.31 *Let* $x(t) = 3\cos(\omega_1 t) + 4\sin(\omega_2 t)$*, where* $\omega_1 = 120\pi$ *and* $\omega_2 = 180\pi$*. The signal* $x(t)$ *is applied to the input of a filter of transfer function* $H(s) = 1/(1 + 120\pi/s)$.

Evaluate the power spectra density $S_y(\omega)$ *of the the signal* $y(t)$ *at the filter output. Evaluate the average power* $\overline{y^2(t)}$ *of* $y(t)$.

Problem 12.32 *A filter which has a transfer function* $H(s) = K/(1 + s/\omega_c)$ *receives an input signal* $x(t) = A\cos(2\pi f_0 t)$*, where* $A = 5$ *Volt and* $f_0 = 500$ *Hz, and produces an output signal* $y(t)$.

a) Let $K = 1$ *and* $\omega_c = 500\pi$ *r/s. Evaluate the average signal power at the filter output.*

b) Let $K = 1$*. Determine the value of* ω_c *so that the average power of the output signal* $y(t)$ *be* 5 *Watt.*

c) Let $\omega_c = 1000\pi$ *r/s. Determine the value of* K *so that the average power of the output signal* $y(t)$ *be* 5 *Watt.*

Problem 12.33 *The periodic signal* $v(t) = \sum_{n=-\infty}^{\infty} (-1)^n \Lambda_{T/4}(t - nT/2)$ *is applied to the input of filter of frequency response* $H(j\omega) = 4\Lambda_{12}(\omega)$ *and output* $y(t)$*. Evaluate*

a) the average power of the signal $v(t)$*,*

b) the average power of $y(t)$ *if* $T = 2\pi/3$*,*

c) the average power of $y(t)$ *if* $T = \pi/6$.

Problem 12.34 *A voltage* $v_E(t)$ *is applied to the input of a first order lowpass RC filter with* $RC = 1$, *of which the output is* $v_S(t)$. *For each of the following cases evaluate the average power of the input and output signal* $v_E(t)$ *and* $v_S(t)$, *respectively.*
a) The power spectral density of $v_E(t)$ *is* $S_{v_E}(\omega) = A[\delta(\omega+1) + \delta(\omega-1)]$.
b) The power spectral density of $v_E(t)$ *is* $S_{v_E}(\omega) = u(\omega+1) - u(\omega-1)$.
c) The power spectral density of $v_E(t)$ *is* $S_{v_E}(\omega) = A$.

Problem 12.35 *The signal* $x(t) = \sin(4\pi t)$ *is applied to the input of a filter of transfer function* $H(s) = 1/(s+1)$ *and output* $y(t)$.
a) Evaluate the power spectral density $S_x(\omega)$ *of the signal* $x(t)$.
b) Evaluate the average power of the signal $x(t)$.
c) Evaluate the normalized energy of one period of the signal $x(t)$.
d) Evaluate the power spectral density $S_y(\omega)$ *of the signal* $y(t)$ *at the filter output.*
e) Evaluate the average power $\overline{y^2(t)}$ *of the filter output signal* $y(t)$.

Problem 12.36 *The signal* $v(t) = \sum_{n=-\infty}^{\infty} \delta(t-12n)$ *is applied to the input of a linear system of impulse response* $h(t) = \sin(\pi t)[u(t) - u(t-12)]$. *Evaluate the power spectral density of the filter output signal* $y(t)$.

Problem 12.37 *Let* $x(t)$ *be a periodic signal of period* 5×10^{-3} *seconds and exponential Fourier series coefficients* X_n, *evaluated with an analysis interval equal to its period, given by*

$$X_n = \begin{cases} 1, & n = \pm 1 \\ \pm j/5, & n = \pm 2 \\ (1 \mp 2j)/10, & n = \pm 4 \\ 0, & \text{otherwise.} \end{cases}$$

The properties of the message $m(t)$ *are* $\overline{m(t)} = 0$ *Volt,* $\overline{m^2(t)} = 2$ *Watt,* $|m(t)|_{max} = 5$ *Volt.* $M(f) = 0$ *for* $|f| > 7.5 \times 10^3$ *Hz.*

For each of the five possible frequency responses of the bandpass filter evaluate the maximum amplitude of the modulated signal $y(t)$.

Defining the Harmonic Distortion Rate HDR as

$$HDR = \frac{P_h}{P_T} \times 100\%$$

where P_h *is the average power of the signal harmonics other than the fundamental and* P_T *is the total signal average power.*
a) Evaluate the HDR of the signal $x(t)$.
b) The signal $x(t)$ *is applied to the input of a filter the transfer function of which is given by*

$$H(s) = \left.\frac{1}{s+1}\right|_{s \to s/(400\pi)}.$$

Evaluate the HDR of the filter output signal $y(t)$.

Problem 12.38 *Let* $x(t) = v(t) + a\,v(t-t_0)$, *where* $v(t)$ *is a power signal and* t_0 *is a constant.*
Show that $\overline{x^2(t)} = (1+a^2)\,\overline{v^2(t)} + 2a\,r_v(t_0)$, *where* $\overline{x^2(t)}$ *is the average power of* $x(t)$, $\overline{v^2(t)}$ *is that of* $v(t)$ *and* $r_v(t_0)$ *is the autocorrelation function of* $v(t)$ *evaluated at* $t = t_0$.

12.20 Answers to Selected Problems

Problem 12.1 a)

i)

$y(t) = \sum_{n=-\infty}^{\infty} \sin\pi(t-11n)\left\{u(t-11n) - u(t-11n-11)\right\}$

$$\begin{aligned} Y(j\omega) &= 2\pi \sum_{n=-\infty}^{\infty} H_n \delta(\omega - n\omega_0) \\ &= -j\pi \sum_{n=-\infty}^{\infty} \left[e^{-jn\pi + j\beta \mathrm{T}/2} Sa(n\pi - \beta T/2) - e^{-jn\pi - j\beta T/2} Sa(n\pi + \beta T/2)\right] \delta(\omega - n\omega_0) \end{aligned}$$

$$\begin{aligned} &Y(j\omega) = \\ &j\pi \sum_{n=-\infty}^{\infty} \left[e^{-jn\pi + j11\pi/2} Sa(n\pi - 11\pi/2) - e^{-jn\pi - j11\pi/2} Sa(n\pi + 11\pi/2)\right] \delta(\omega - n2\pi/11) \end{aligned}$$

ii)

$$Y(j\omega) = -j\pi\left\{\delta(\omega - \pi) - \delta(\omega + 6)\right\}$$

$$Y_n = \left\{\begin{array}{cl} \mp j/2\ , & n = \pm 6 \\ 0 \quad , & n \neq \pm 6 \end{array}\right\}$$

$$b) \quad h(t) = \sin\pi t\left\{u(t) - u(t-12)\right\}$$

b) $h(t) = \sin\pi t\left\{u(t) - u(t-12)\right\}$.

$$\varepsilon_h(t) = (T^2/4)\left|e^{-j(\omega-\pi)\mathrm{T}/2}\ Sa\left\{(\omega-\pi)\,\mathrm{T}/2\right\} -\ e^{-j(\omega+\pi)T/2} Sa\left\{(\omega+\pi)\,T/2\right\}\right|^2$$

$$y(t) = \sin\pi t.$$

$$S_y(\omega) = (\pi/2)\left\{\delta(\omega - \pi) + \delta(\omega + \pi)\right\}$$

Problem 12.2

a) The signal, having an impulsive spectrum, is periodic. b)

$$\begin{aligned} S_f(\omega) = 98\pi\delta(\omega) &+ 18\pi\left\{\delta\left(\omega - 2\pi\times 10^3\right) + \delta\left(\omega + 2\pi\times 10^3\right)\right\} \\ &+ 2\pi\left\{\delta\left(\omega - 8\pi\times 10^3\right) + \delta\left(\omega + 8\pi\times 10^3\right)\right\} \end{aligned}$$

c)

$$P = \overline{f^2(t)} = \sum_{n=-\infty}^{\infty} |F_n|^2 = 49 + 2\times 9 + 2\times 1 = 69$$

d)

$$P = \frac{1}{T}E, \quad E = TP = \frac{2\pi}{\omega_0}\times 69 = 69\times 10^{-3}$$

e)

$$G_n = \begin{cases} \pm j\,3\,K, & n = \pm 1 \\ 0, & n \neq \pm 1 \end{cases}$$

$$\overline{g^2(t)} = \sum_{n=-\infty}^{\infty} |G_n|^2 = 2\times 9\ K^2 = 18\ K^2$$

Problem 12.5

a)

$\varepsilon_{xx}(\omega) = \frac{2}{1+\omega^2}$

b) $E = 1$

$$E = 1$$

c) $E = 0.4373A^2/\pi$

Problem 12.6

$$Y_n = \begin{cases} \frac{\mp j\beta}{2(100\,\pi \pm j\beta)}, & n = \pm 1 \\ 0\ , & otherwise \end{cases}$$

$$S_y(\omega) = 2\pi \sum_{n=-\infty}^{\infty} |Y_n|^2 \delta(\omega - n\omega_0) = 2\pi \times 0.1475 \{\delta(\omega - \beta) + \delta(\omega + \beta)\}$$

$\overline{y^2(t)} = 0.295$

Problem 12.7

See Fig. 12.22

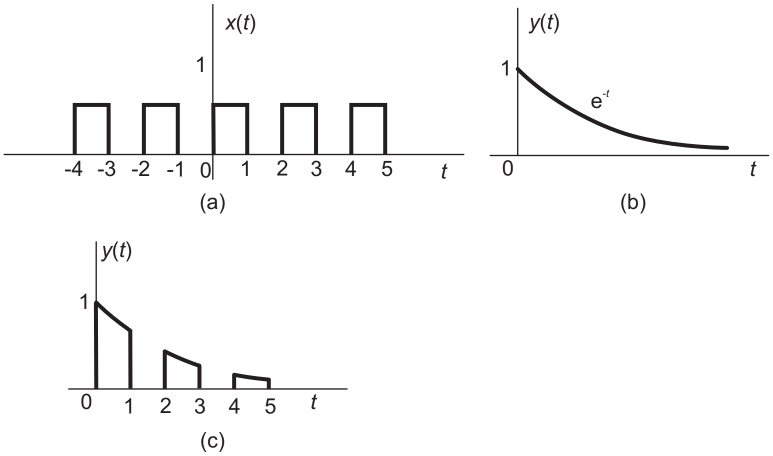


FIGURE 12.22

Figure for Problem 12.7

a) $E_x = \infty E_y(t) = 1/2$ Joules

b) The average normalized powers are $\overline{x^2(t)} = (1/2) \cdot 1 = 1/2$ Watt.

$\overline{y^2(t)} = 0.$

$y(t)$ is an energy signal since $E_y < \infty$, $z(t)$ is periodic since $x(t)$ is periodic. The signal $z(t)$ is therefore a power signal.

Problem 12.8

a)

$$H_{LP}(s) = \frac{K}{s^2 + 1.4142\ s + 1}$$

$$H_{BP}(s) = \frac{K\ s^2}{(s^2+1)^2 + 1.4142\,s\,(s^2+1) + s^2}$$

$K = 1.$

b)

$$\omega_L = 1.6180$$

c)

$$|H_{BP}(j\omega_0)| = 5.01$$

$y(t)$ a sinusoid of amplitude $A = 35.07$, average normalized power 614.95 Watt.

Problem 12.9

a)

$\overline{v^2(t)} = 6.5$ Watt.

b) *Energy signal, being of finite duration
$E_w = \int_0^4 \left(t^2/16\right) dt = 1.333$ joules
c) $E_x = 11\ E_w = 14.63$ joules
d) $\overline{y^2(t)} = \frac{1}{5} E_w = 0.267$ Watt.

Problem 12.10

a) $\overline{x(t)} = 0$ since $X(j\omega)$ has no impulse at the origin $\omega = 0$.
b) $x(t)$ is not periodic. To be periodic the spectrum has to be composed solely of impulses.
c) See Fig.12.23

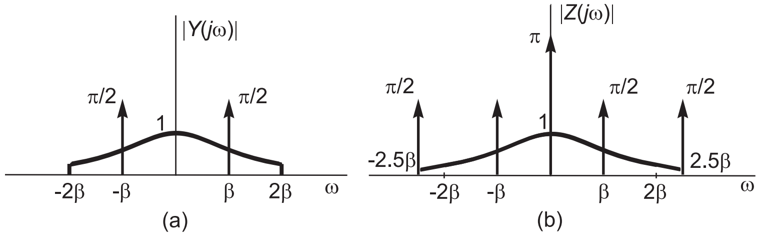


FIGURE 12.23
Figure for Problem 4.10

Problem 12.11

a) Total Energy $=A^2/2$ watt. Power signal
b) Total Energy $=A^2/2000$ Joule. Average normalized power = 0. Energy signal [equal to a single period of $v(t)$].
c) $\overline{x^2(t)} = \frac{1}{6}\left(1 - e^{-6}\right) = 0 \cdot 15$. Power signal. Energy $= \infty$
d) $\overline{z^2(t)} = A^2$, Power signal. Total Energy $=\infty$.

Problem 12.12

a)

$$\overline{y^2(t)} = 2.5 \text{ Watt}$$

b) Note that the average power of a sinusoid of Amplitude A is $A^2/2$

$$\omega_c = 2565.1 \text{ r/s}$$

c)

$$K = 0.8944$$

Problem 12.13

a)

$$V(j\omega) = 5\ \text{Sa}(0.25\omega)\, e^{-j0.25\omega}$$

$$F(j\omega) = 30\pi\delta(\omega) - 10e^{-j7\pi/6}\delta(\omega - \pi) + 10e^{j7\pi/6}\delta(\omega + \pi)$$

$$F_n \begin{cases} 15, n = 0 \\ \mp(5/\pi)\, e^{\mp j7\pi/6}, n = \pm 1 \\ 0, \text{otherwise} \end{cases}$$

b)

$$\varepsilon_v(\omega) = |V(j\omega)|^2 = 25\ \text{Sa}^2(0.25\omega)$$

$$P = \overline{f^2(t)} = 230.07 \text{ Watt}$$

Problem 12.14

$$\overline{y(t)} = \overline{f(t)}\, H(0) = 15 \frac{25\pi}{7^2 + (5\pi)^2} = 3.984$$

Problem 12.15

a)

$$X_n = 1,\ -0.9,\ 0.636,\ -0.301,\ 0,\ 0.18$$

for $n = 0,\ \pm 1,\ \pm 2,\ \pm 3,\ \pm 4,\ \pm 5$ respectively, and $X_n = 0$, otherwise.b)

See Fig. 12.24.

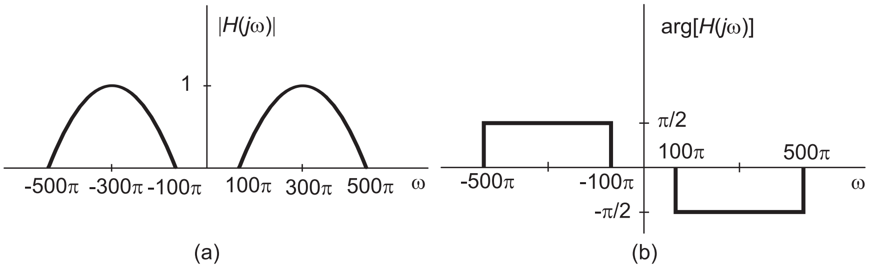


FIGURE 12.24

Amplitude and phase of frequency response, Problem 12.15

c)

$$Y_2 = \mp j0.4775, \quad Y_3 = \pm j0.3001, \quad Y_5 = \mp j0.135, \quad Y_n = 0, \text{ otherwise.}$$

d)

$$y(t) = 0.955 \sin 200\pi t - 0.6 \sin 300\pi t + 0.27 \sin 500\pi t$$

Problem 12.16

a) See Fig. 12.25

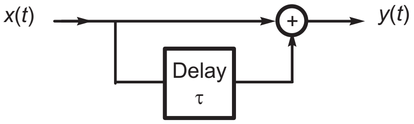


FIGURE 12.25

Figure for Problem 12.16

b)

$$h(t) = \delta(t) + \delta(t - \tau), \quad H(j\omega) = 1 + e^{-j\omega\tau}$$

c)

$$S_x(\omega) = 2\pi \left\{2.5^2 \delta(\omega - 2000\pi) + 2.5^2 \delta(\omega + 2000\pi)\right\}$$

d)

$$S_y(\omega) = 2\pi \times 2.387 \left\{\delta(\omega - 2000\pi) + \delta(\omega + 2000\pi)\right\}$$

$$\overline{y^2(t)} = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_y(\omega)\, d\omega = 2 \times 2.387 = 4.775 \text{ Watt}$$

See Fig. 12.26

Problem 12.17

a)

$$\varepsilon_x(\omega) = \frac{1}{\omega^2 + 49}$$

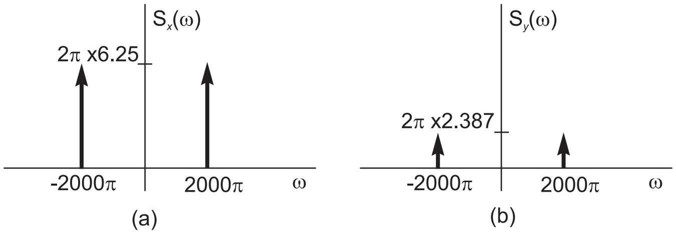


FIGURE 12.26
Figure for Problem 12.16

b)

$$\varepsilon_y(\omega) = \begin{cases} \dfrac{25}{\omega^2+49}, & 1.1 \leqslant \omega \leqslant 1.3 \\ 0, & \text{otherwise} \end{cases}$$

Problem 12.18

$$S_{vv}(\omega) = \pi\left[\delta(\omega - W/4) + \delta(\omega + W/4)\right]$$
$$S_{yy}(\omega) = (15\pi/16)\left[\delta(\omega - W/4) + \delta(\omega + W/4)\right]$$

$$\overline{y^2(t)} = 15/16 = 0.9375 \text{ Watt}$$

Problem 12.19
a) See Fig. 12.27

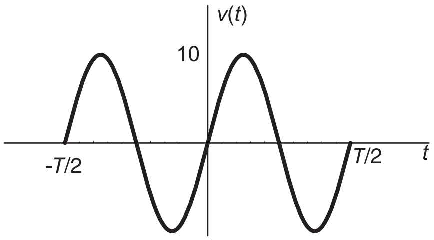


FIGURE 12.27
Figure for Problem 12.19

$E = 50T, \quad P = 0, \quad \varepsilon_v(\omega) = |V(j\omega)|^2.$

$$\varepsilon_v(\omega) = 25T^2\left\{Sa^2\left[T(\omega - \beta/2)\right] - 2Sa\left[T(\omega - \beta/2)\right]Sa\left[T(\omega+\beta)/2\right]\right\}$$
$$+25T^2\left\{Sa^2\left[T(\omega+\beta)/2\right]\right\}.$$

b) $100\pi T$

Problem 12.20

$$V_n = \begin{cases} 5, & n = \pm 1 \\ \mp j2.5, & n = \pm 4 \\ 4, & n = \pm 10 \end{cases}$$

$$S_n = |V_n|^2 = \begin{cases} 25, & n = \pm 1 \\ 6.25, & n = \pm 4 \\ 16, & n = \pm 10 \end{cases}$$

Problem 12.22

a)

For $-t_0 \le -t + b - t_0 \le b - t_0$ i.e. $0 \le t \le b$

$$r_{vv}(t) = -t + b - t_0 + t_0 = b - t$$

For $-t_0 \le -t - t_0 \le b - t_0$ i.e. $-b \le t \le 0$

$$r_{vv}(t) = b - t_0 + t + t_0 = b + t$$

b)

$$R_{vv}(j\omega) = b^2 Sa^2(b\omega/2)$$

c)

$$\varepsilon(\omega) = R_{vv}(j\omega).$$

$$E = b \text{ joules}$$

Problem 12.23

a)

$$|X(j\omega)|^2 = \left(1 - 2e\cos(\omega) + e^2\right) / \left(1 + \omega^2\right)$$

b)

$|Y(j\omega)|^2 = \frac{1}{\omega^4+4}$

Problem 12.24

a) Energy : $\int_0^{+\infty} \left(e^{-t}\right)^2 dt = 0.5.$

b) $V(j\omega)\, 1/\left(1 + \omega^2\right)$

Energy= $\frac{1}{2\pi} \int_{-2\pi}^{+2\pi} \frac{1}{1+\omega^2} d\omega = \frac{1}{2\pi} \left[\tan^{-1}(\omega)\right]_{-2\pi}^{+2\pi} = 0.45$

Problem 12.25

a) Energy signal.

b)

The energy spectral density is $1/\left(1 + \omega^2\right)$

c)

0.25.

d)

0.45.

e)

$$r_{vv} = 0.5e^{-t}u(t) + 0.5e^{+t}u(-t)$$

f)

$$\mathcal{F}\{r_{vv}(t)\} = 1/\left(1 + \omega^2\right)$$

Problem 12.26

a) $E_v = 4.$

b)

$E_y = 2$

c)

$E_y = 0.383.$

d)

$E_y = 0.5$

e)

$E_y = 4.$

Problem 12.27

a)

$E = 45T_a$ Joule.

b)

$E = 22.5T_a$ Joule. $P = 22.5T_a/15T_a = 1.5$ Watt c)

$P = 1$ Watt.

Problem 12.28

a)

$$E = 0.5 + (4 \times 0.5) + 0.5 = 3$$

b)

$$S_s(\omega) = 0.637\left[\delta(\omega + \pi) + \delta(\omega - \pi)\right]$$

Problem 12.29

5Sol 51

a)

$P = 17$

b) $P = 0.75$

c)

$P = 3$

d)

$P = 33.33$

Problem 12.30

a) $\overline{z^2(t)} = 3$. b) $\overline{v^2(t)} = 5$.

Problem 12.31

$$\begin{gathered} S_y(\omega) = 2\pi \times (9/8)\left[\delta(\omega + 120\pi) + \delta(\omega - 120\pi)\right] \\ +2\pi \times (36/13)\left[\delta(\omega + 180\pi) + \delta(\omega - 180\pi)\right] \\ \overline{y^2(t)} = 7.8 \end{gathered}$$

Problem 12.32

Sol 54

a) $\overline{x^2(t)} = 2.5$. b) $\overline{y^2(t)} = 0.4$. $\omega_c = 2565$ r/s. c) $\overline{y^2(t)} = 0.4.K = 0.894$.

Applications of Angle Modulation

i. it is used for commercial radio broadcasting, TV sound transmission, cellular radio, microwave and satellite communication systems.

2. AMPLITUDE MODULATION

Amplitude modulation is the process of changing the amplitude of the carrier signal inaccordance with the amplitude of the modulating signal. Frequency and phase of the carrier signal are not altered during this process.

Let the modulating signal and carrier signal can be written as

$$v_m(t) = V_m \text{Sin}\ \omega_m t \qquad \text{.... 3}$$

$$v_C(t) = V_C \text{Sin}\ \omega_C t \qquad \text{.... 4}$$

According to the definition, the amplitude of the carrier signal is changed after modulation,

$$V_{AM} = V_C + v_m(t) = V_C + V_m \text{Sin}\ \omega_m t \qquad \text{....5}$$

$$= V_C\left[1 + \frac{V_m}{V_C}.\text{Sin}\omega_m t\right] = V_C\ (1 + m_a \text{Sin}\ \omega_m t) \qquad \text{... 6}$$

$m_a = V_m/V_C =$ *"modulation index or depth of modulation"*

3. AM ENVELOPE

The shape of the modulated signal is defined as AM envelope, because, it contains all frequencies that make up the AM signal and it used to communicate the information through the system.

The instantaneous amplitude of modulated signal or AM envelope can be written as

$$v_{AM}(t) = V_{AM} \text{Sin}\ \omega_C t \qquad \text{.... 7}$$

Substitute the value of V_{AM} in equation 7

$$v_{AM}(t) = V_C (1 + m_a \text{Sin}\ \omega_m t)\ \text{Sin}\ \omega_C t$$

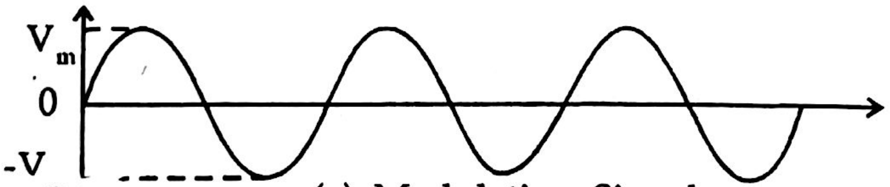
$$= V_C \text{Sin}\ \omega_C t + m_a V_C \text{Sin}\ \omega_m t\ .\ \text{Sin}\ \omega_C t \qquad \text{.... 8}$$

we know $$\sin \omega_m t \sin \omega_C t = \frac{\cos(\omega_c - \omega_m)t - \cos(\omega_c + \omega_m)t}{2}$$

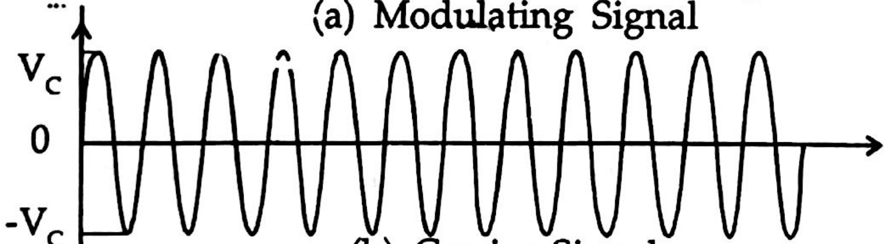
$$v_{AM}(t) = V_C \sin \omega_C t + \frac{m_a V_C}{2} [\cos(\omega_C - \omega_m)t - \cos(\omega_C + \omega_m)t]$$

$$\boxed{v_{AM}(t) = V_C \sin \omega_C t + \frac{m_a V_C}{2} [\cos(\omega_C - \omega_m)t - \cos(\omega_C + \omega_m)t]} \quad ...9$$

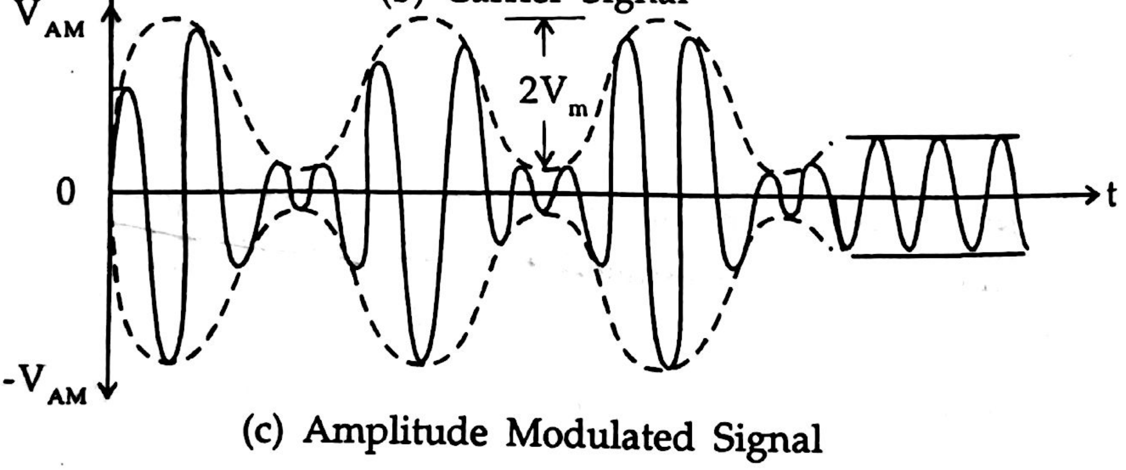
Figure 1 shows the graphical representation of amplitude modulation wave. It clearly shows that the amplitude of the carrier is varied in accordance with the modulating signal while frequency of carrier wave remains the same.



(a) Modulating Signal



(b) Carrier Signal



(c) Amplitude Modulated Signal

Figure 1 : Graphical representation of AM

It is important to note that,

i. if message signal is absent, the output is simply the carrier signal.

ii. The shape of the envelope is identical to shape of the modulating signal.

SE4ECD1154.

4. AM FREQUENCY SPECTRUM AND BANDWIDTH

The equation (9) of an amplitude modulated wave contains three terms. The 1st term of R.H.S. represents the carrier wave. The 2nd and 3rd terms are identical which are called as ***"lower side band (LSB) and upper side band (USB)"***.

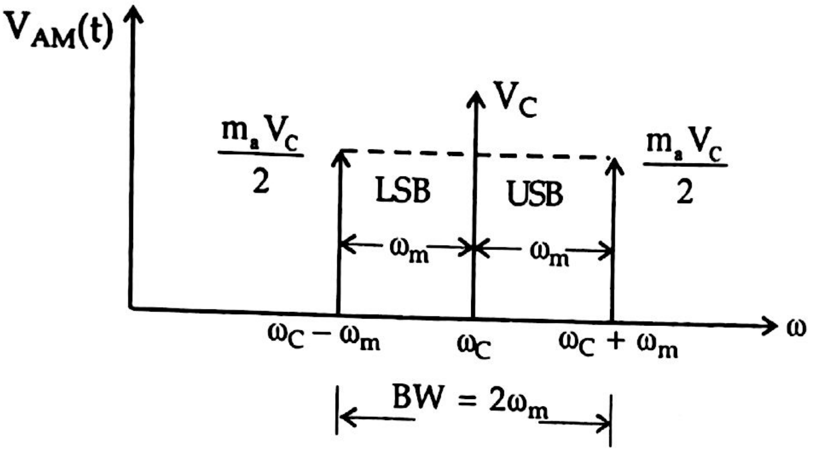


Figure 2 : Frequency spectrum of AM with carrier

Figure 2 shows the frequency spectrum of AM. It shows that two side band terms lying on either sides of carrier term which are separated by ω_m. The range of frequency between $(\omega_C - \omega_m)$ is known as LSB and $(\omega_C + \omega_m)$ is known as USB. The spacing between these two bands with respect to carrier is ω_m. The bandwith of AM can be determined by using these side bands. Hence ***"BW is twice the frequency of modulating signal"***.

5. PHASOR REPRESENTATION OF AM WITH CARRIER

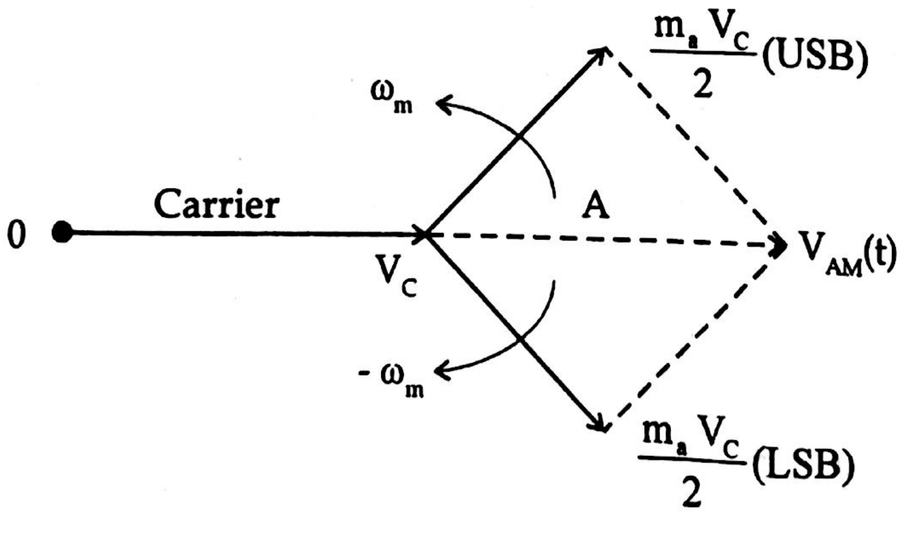


Figure 3 : Phasor representation of AM

Figure 3 shows the phasor representation of AM with carrier. It is the easy way of representation of AM, where V_C is carrier wave phasor, taken as reference phasor. The two sidebands having a frequency of $(\omega_C + \omega_m)$ and $(\omega_C - \omega_m)$ are represented by two phasors rotating in opposite directions with angular frequency of ω_m. The net or resultant phasor is $V_{AM}(t)$ the vector sum of two side bands with carrier. It depends on the position of the sideband phasor and carrier wave phasor.

That is the phasors for the carrier and LSB and USB combine sometimes or some time subtracts. The maximum positive amplitude of the envelope occurs if the carrier, LSB and USB all are have positive values or in phase $(V_{maz} = V_C + V_{LSB} + V_{USB})$. The minimum positive amplitude of envelope occcurs if the carrier and the side bands are in out of phase $V_{min} = V_C - V_{LSB} - V_{USB}$ as shown in figure 3.

6. COEFFICIENT OF MODULATION OR PERCENT MODULATION OR MODULATION INDEX

The modulation index used to describe the amount of amplitude change occured in AM envelopes. It can be computed as follows.

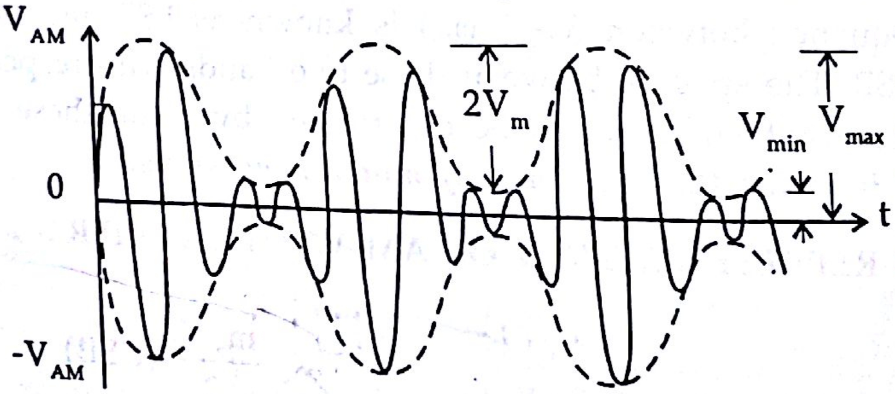


Figure 3(a) : Amplitude modulated signal

From figure 3(a), $2V_{(modulating)max} = V_{max} - V_{min}$

$$V_m = V_{(modulating)\ max} = \frac{V_{max} - V_{min}}{2} \quad \ldots 10$$

and $V_{(carrier)max} = V_{max} - V_m$

$$= V_{max} - \left(\frac{V_{max} - V_{min}}{2}\right) = \frac{V_{max} + V_{min}}{2} \quad \ldots 11$$

Therefore $$m_a = \frac{V_m}{V_C} = \frac{(V_{max} - V_{min})/2}{(V_{max} + V_{min})/2} = \frac{V_{max} - V_{min}}{V_{max} + V_{min}} \quad \ldots 12$$

$$\therefore \quad \boxed{\% \; m_a = \frac{V_{max} - V_{min}}{V_{max} + V_{min}} \times 100} \quad \ldots 13$$

7. DEGREES OF MODULATION

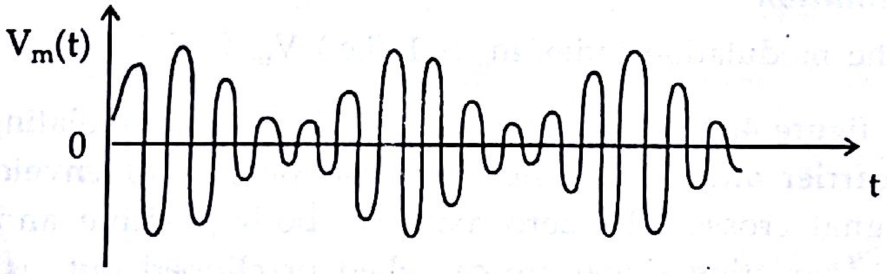
The modulating signal is preserved in the envelope of amplitude modulated signal only if $V_m < V_C$, then $m_a < 1$

Where V_m = maximum amplitude of modulating signal

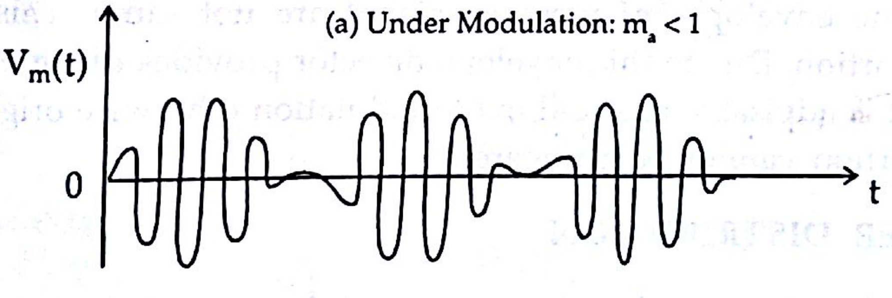
V_C = maximum amplitude of carrier signal.

There are three degrees of modulation depending upon the amplitude of the message signal relative to carrier amplitude.

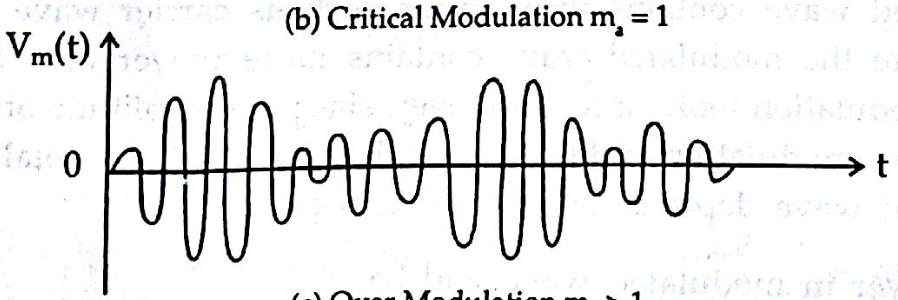
i) Under modulation ii) Critical modulation iii) Over modulation



(a) Under Modulation: $m_a < 1$



(b) Critical Modulation $m_a = 1$



(c) Over Modulation $m_a > 1$

Figure 4 : Three degrees of modulation

i). Under modulation

In this case the modulation index $m_a < 1$ (i.e.) $V_m < V_C$ . . . 14

It is shown in figure 4(a). Here the envelope of amplitude modulated signal does not reach the zero amplitude axis. Hence the message signal is fully preserved in the envelope of the AM wave. This is known as ***"under modulation."*** An envelope detector can recover the message signal without any distortion.

ii). Critical Modulation

In this case the modulation index $m_a = 1$ (i.e.) $V_m = V_C$ . . . 15

It is shown in figure 4(b). Here the envelope of the modulated signal just reaches the zero amplitude axis. The message signal remains preserved. This is known as ***"critical modulation."*** In this case also the modulated signal can be recovered by using an envelope detector without any distortion.

iii). Over Modulation

In this type, the modulation index $m_a > 1$ (i.e,) $V_m > V_C$ . . . 16

It is shown in figure 4(c). In this case the amplitude of modulating signal is greater than carrier amplitude. Therefore that portion of envelope of the modulated signal crosses the zero axis. So, both positive and negative extensions of modulating signal are cancelled or clipped out, as shown in figure 4(c). The envelopes of message signal are not same. This is called **envelope distortion.** Due to this envelope detector provides distorted message signal. Thus it is advisable to avoid over modulation otherwise original signal without distortion cannot be recovered.

8. AM POWER DISTRIBUTION

The modulated wave contains three terms such as carrier wave, LSB, and USB. Therefore the modulated wave contains more power than the carrier had before modulation took place. Moreover, since the amplitude of sidebands depend on the modulation index, it is anticipated that the total power in the modulated wave depends on the modulation index.

The total power in modulated wave will be

$$P_t = P_c + P_{LSB} + P_{USB} \quad \ldots 17$$

$$P_t = \frac{V_{carrier}^2}{R} + \frac{V_{LSB}^2}{R} + \frac{V_{USB}^2}{R} \qquad \ldots 18$$

Where, $V_{carrier}$ = RMS value of carrier voltages. . . . 19

V_{LSB} = V_{USB} = RMS value of upper and lower side band voltages

R = Resistance in which power is dissipated

$$P_{carrier} = \frac{V_{carrier}^2}{R} = \frac{\left(V_C/\sqrt{2}\right)^2}{R} = \frac{V_C^2}{2R} \qquad \ldots 20$$

Similarly,

$$P_{LSB} = P_{USB} = \frac{V_{SB}^2}{R} = \frac{\left(\dfrac{\frac{m_a V_C}{2}}{(\sqrt{2})}\right)^2}{R} = \frac{m_a^2 V_C^2}{8R} \qquad \ldots 21$$

V_C = maximum amplitude of carrier wave

V_{SB} = $\frac{m_a V_C}{2}$ = maximum amplitude of side bands.

We know

$$P_t = P_C + P_{LSB} + P_{USB} = \frac{V_C^2}{2R} + \frac{m_a^2 V_C}{8R} + \frac{m_a^2 V_C{}^2}{8R} \qquad \ldots 22$$

Therefore

$$P_t = \frac{V_C^2}{2R} + \frac{m_a^2 V_C{}^2}{4R} = \frac{V_C^2}{2R}\left[1 + \frac{m_a^2}{2}\right] \qquad \ldots 23$$

We know that

$$P_C = \frac{V_C^2}{2R}.$$

thus

$$P_t = P_C\left[1 + \frac{m_a^2}{2}\right] \qquad \ldots 24$$

or

$$\boxed{\frac{P_t}{P_C} = \left[1 + \frac{m_a^2}{2}\right]} \qquad \ldots 25$$

If $m_a = 1$; i.e. for 100 % modulation

then $\frac{P_t}{P_c} = 1.5$ or $\boxed{P_t = 1.5\,P_C}$ . . . 26

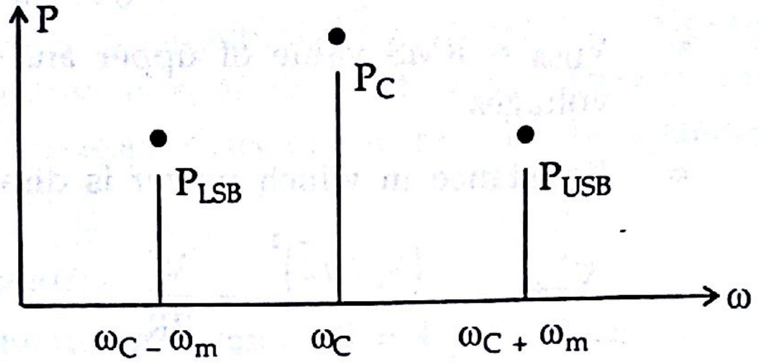


Figure 5 : Power spectrum of AM

The power spectrum of AM is shown in figure 5 it is important to note that, if $m_a = 1$, the maximum power is the side band is equal to only one fourth of the power in the carrier, it proves that the most of the power us wasted in the carrier.

9. AM CURRENT RELATION AND EFFICIENCY

From equation 25 we get $P_t = P_C\left[1+\frac{m_a^2}{2}\right]$

We know $P_t = I_t^2.R$ and $P_C = I_C^2.R$

Hence, $I_t^2 = I_C^2\left[1+\frac{m_a^2}{2}\right]$ or $\boxed{I_t = I_C\sqrt{\left[1+\frac{m_a^2}{2}\right]}}$ . . . 27

Where I_t = total or modulated current; I_C = carrier current

% Efficiency

It can be defined as the ratio of power in sidebands to total power, because side bands only contain the useful information.

$$\%\eta = \frac{\text{Power in side band}}{\text{Total power}} \times 100 \quad \ldots 28$$

$$\%\eta = \frac{P_{LSB} + P_{USB}}{P_{total}} \times 100 = \frac{\frac{m_a^2 V_C^2}{8R} + \frac{m_a^2 V_C^2}{8R}}{\frac{V_C^2}{2R}\left[1+\frac{m_a^2}{2}\right]} \times 100 \qquad \ldots 29$$

$$= \frac{\frac{m_a^2 V_C^2}{4R}}{\frac{V_C^2}{2R}\left[1+\frac{m_a^2}{2}\right]} \times 100 = \frac{\frac{m_a^2 P_C}{2}}{P_C\left[1+\frac{m_a^2}{2}\right]} \times 100 \qquad \ldots 30$$

$$\therefore \boxed{\%\ \eta = \frac{m_a^2}{2+m_a^2} \times 100} \qquad \ldots 31$$

If, $m_a = 1$ then $\%\ \eta = \frac{1}{3} \times 100 = 33.3\%$. . . . 32

From this we conclude that only 33.3% of energy is used and remaining power is wasted by the carrier transmission along with the sidebands.

10. MODULATION BY SEVERAL SINE WAVES OR COMPLEX INFORMATION SIGNAL

In the previous section, AM for singne message signal were analysed. In practice, modulation of a carrier by several sine waves simultaneously are needed.

Let V_1, V_2, V_3 ..... etc., be the simultaneous modulation voltages with frequencies f_{m1}, f_{m2}, f_{m3} . . . . Then the total modulating voltage V_t will be equal to the square root of sum of square of individual voltages.

$$V_{Am}(t) = V_m \sin\omega_C t + V_C \frac{m_t}{2} \{[\cos(\omega_C - \omega_{m1})t - \cos(\omega_C + \omega_{m2})t] + [\cos(\omega_C - \omega_{m2})t - \cos(\omega_C + \omega_{m2})t] + \ldots\}$$

Let $V_t = \sqrt{V_1^2 + V_2^2 + \ldots.}$ . . . 33

Dividing both sides by V_C we get

$$\frac{V_t}{V_C} = \sqrt{\frac{V_1^2 + V_2^2 + \ldots.}{V_C^2}} + \ldots\ldots = \sqrt{\frac{V_1^2}{V_C^2} + \frac{V_2^2}{V_C^2} + \ldots\ldots} \qquad \ldots 34$$

$$m_t = \sqrt{m_1^2 + m_2^2 + \ldots\ldots} \quad \ldots 35$$

Modulation index can also be found out by using the above principle that, while modulating the carrier simultaneously using several sine waves, the carrier power will be unaffected, but the total side band power will be the sum of individual side band powers. Hence

$$P_{SB} = P_{SB1} + P_{SB2} + \ldots\ldots. \quad \ldots 36$$

We know that $P_{SB} = \frac{P_C m_a^2}{2}$ By substituting this in equation .36, we get

$$\frac{P_C m_a^2}{2} = \frac{P_C m_1^2}{2} + \frac{P_C m_2^2}{2} + \ldots. \quad \ldots 37$$

$$m_{at}^2 = m_1^2 + m_2^2 + \ldots..$$

$$m_{at} = \sqrt{m_1^2 + m_2^2 + \ldots..} \quad \ldots 38$$

The total modulation index must be less than unity. If m_t is greater than unity, over modulation occurs results in distortion in the output, hence care must be taken at the transmitter side to avoid this problem.

\\/ 11. DOUBLESIDE BAND SUPPRESSED CARRIER AM (DSB-SC-AM)

i) Two important parameters of a communication system are transmitting power and the bandwith. Hence saving of power and bandwidth are highly desirable in a communication system.

ii) In, AM with carrier scheme, there is wastage in both transmitted power and the bandwidth. In order to save the power in amplitude modulation the carrier may be suppressed, because it does not contain any useful information. This scheme is called as the **Double Side Band Suppressed Carrier Amplitude Modulation** (DSB - SC - AM). It contains only LSB and USB terms, resulting that a transmission bandwith is twice the frequency of the message signal.

Let the modulating signal $v_m(t) = V_m \sin \omega_m t$

and the carrier signal $v_C(t) = V_C \sin \omega_C t$

When multiplying both the carrier and message signal, the resultant signal is the DSB - SC AM signal

$$v(t)_{DSBSC} = V_m(t)V_C(t)$$

Therefore $$v(t)_{DSB-SC} = V_m \sin \omega_m t \,.\, V_C \sin \omega_C t \qquad \ldots 39$$

$$= V_m \,.\, V_C \sin \omega_m t \,.\, \sin \omega_C t$$

$$\boxed{v(t)_{DSBSC} = \frac{V_m V_C}{2}\left[\cos(\omega_C - \omega_m)t - \cos(\omega_C + \omega_m)t\right]} \ldots 40$$

In this case the product of $v_C(t)$ and $v_m(t)$ produces the DSB-SC-AM signal thus, we require product modulator to generate DSB SC signals.

We know that,

$$v_{AM}(t) = V_C \sin \omega_C t + \frac{m_a V_C}{2}\left[\cos(\omega_C - \omega_m)\ t - \cos(\omega_C + \omega_m)t\right] \ldots 41$$

When the equation.41 is compared with equation 40 the unmodulated carrier terms V_C Sin $\omega_C t$ is missing and only two side bands are present, hence the equation (40) is called as DSB - SC - AM.

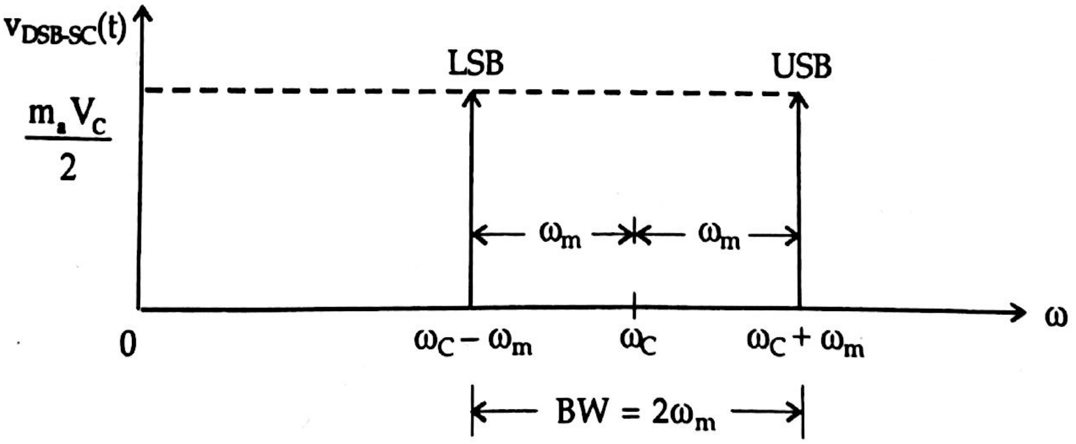


Figure 6 : Frequency spectrum of DSB - SC - AM

Figure 6 shows the frequency spectrum of DSB - SC - AM. It shows that carrier term ω_C is suppressed. It contains only two side band terms having the frequency of ($\omega_C - \omega_m$) and ($\omega_C + \omega_m$). Hence this scheme is known as DSB - SC - AM.

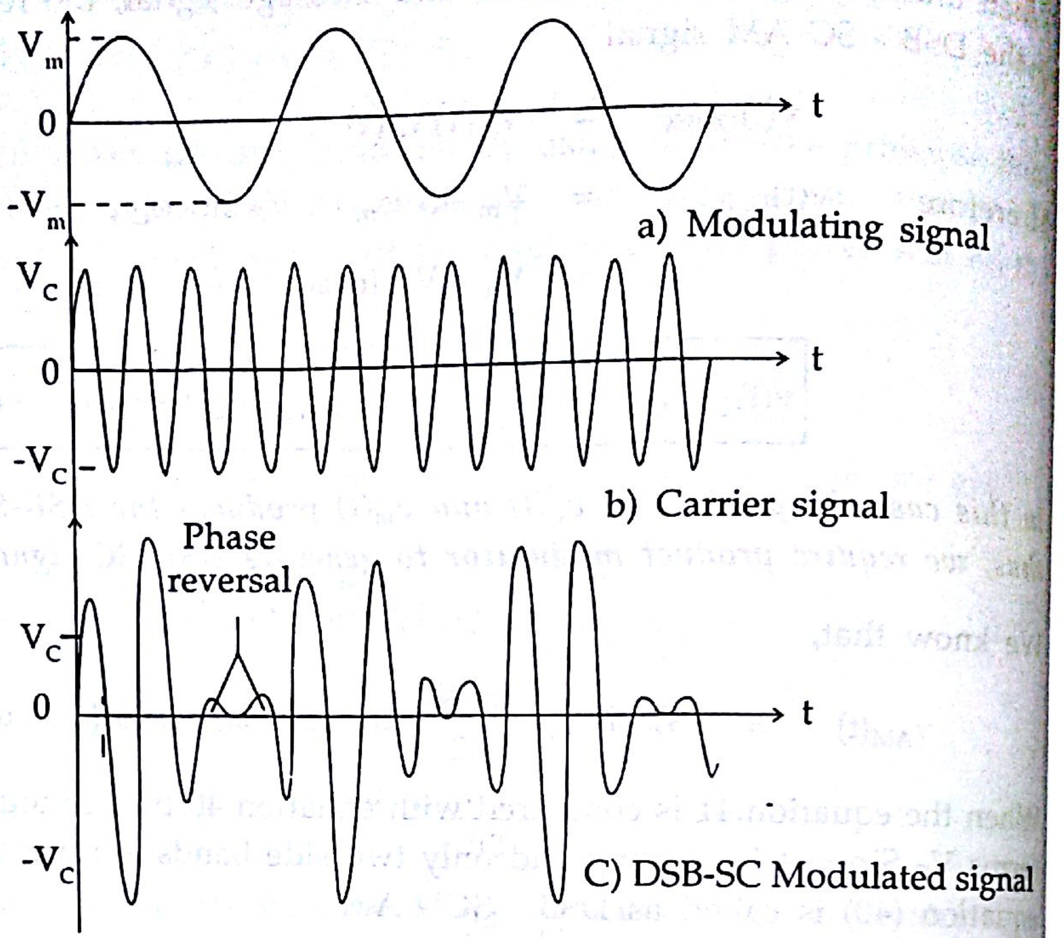


Figure 7 : Graphical Representation of DSB - SC - AM

Figure 7 shows the graphical representation of DSB - SC AM, it exhibits the phase reversal at zero crossing.

Phasor diagram of DSB - SC - AM

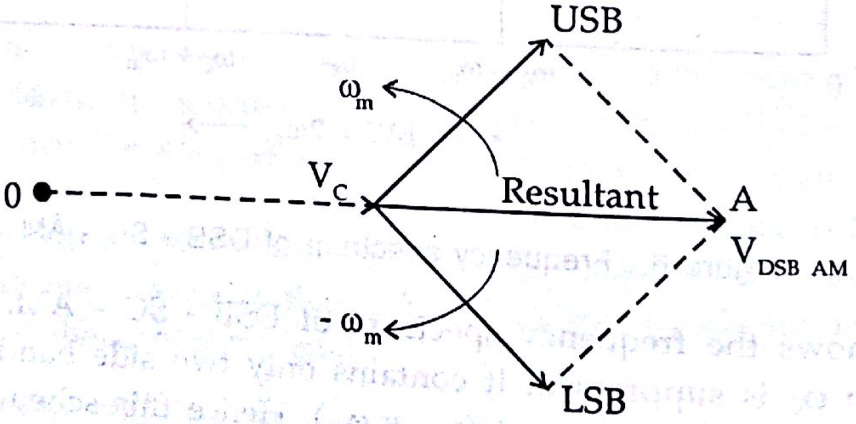


Figure 8 : Phasor diagram of DSB-SC-AM

Assume that the coordinate system rotates in anticlockwise direction at an angular frequency of ω_m. Let us assume the carrier phasor is the reference phasor and oriented in horizontal direction as shown in figure 8 by the dotted line. (because it is suppressed after modulation).

The USB term $\frac{m_a V_C}{2}\cos(\omega_C + \omega_m)t$ rotates at an angular frequency of ω_m in anticlockwise direction and the LSB term $\frac{m_a V_C}{2}\cos(\omega_C - \omega_m)t$ rotates at an angular frequency of ω_m in clockwise direction. Hence the resultant amplitude of the modulated wave at any point is the vector sum of the two side bands.

Power Calculation

We know that, the total power transmitted in AM is

$$P_t = P_{carrier} + P_{LSB} + P_{USB}$$

$$= \frac{V_C^2}{2R} + \frac{m_a^2 V_C^2}{8R} + \frac{m_a^2 V_C^2}{8R} = \frac{V_C^2}{2R} + \frac{m_a^2 V_C^2}{4R} \qquad \ldots 42$$

$$= \frac{V_C^2}{2R}\left[1 + \frac{m_a^2}{2}\right] = P_C\left[1 + \frac{m_a^2}{2}\right] \qquad \ldots 43$$

Where $P_C = \frac{V_C^2}{2R}$

If the carrier is suppressed, then the total power transmitted in DSB-SC-AM is

$$P_t' = P_{LSB} + P_{USB} \qquad \ldots 44$$

We know that,

$$P_{LSB} = P_{USB} = \frac{m_a^2 V_C{}^2}{8R} \qquad \ldots 45$$

Therefore $P_t' = \frac{m_a^2 V_C{}^2}{8R} + \frac{m_a^2 V_C{}^2}{8R} = \frac{m_a^2}{2}\left[\frac{V_C{}^2}{2R}\right]$

$$P_t' = \frac{m_a^2}{2} P_C \qquad \ldots 46$$

$$\text{Power saving} = \frac{P_t - P_t'}{P_t} \quad \ldots 47$$

$$= \frac{\left[1+\frac{m_a^2}{2}\right] P_C - \left(\frac{1}{2} m_a{}^2 P_C\right)}{\left[1+\frac{m_a^2}{2}\right] P_C} = \frac{P_C}{\left[1+\frac{m_a^2}{2}\right] P_C}$$

$$\% \text{ Power saving} = \frac{1}{\left[1+\frac{m_a^2}{2}\right]} \times 100 = \frac{2}{2+m_a^2} \times 100 \quad \ldots 48$$

If $m_a = 1$ then power saving $= \frac{2}{3} \times 100 = 66.7\%$ (i.e) 66.7% of power is saved.

Due to the suppression of the carrier wave, the power saving is increasing from 33.3% to 66.7%

5 12. SINGLE SIDE BAND SUPPRESSED CARRIER AM (SSB - SC - AM)

- In AM with carrier both the transmitting power and bandwidth are wasted, hence, the DSB - SC - AM scheme has been introduced in which power is saved by suppressing the carrier component but the bandwith remains the same (i.e. B.W. = $2\omega_m$).
- Further increase in the saving of power is possible by eliminating one side band in addition to the carrier component, because the USB and LSB are uniquely related by symmetry about the carrier frequency so either one sideband is enough for transmitting as well as recovering the useful message.
- In addition to that, transmission bandwidth can be cut into half if, one side band is suppressed along with the carrier. This scheme is known as *"SSB - SC - AM"*. The block diagram of SSB - SC - AM is shown in figure 9.

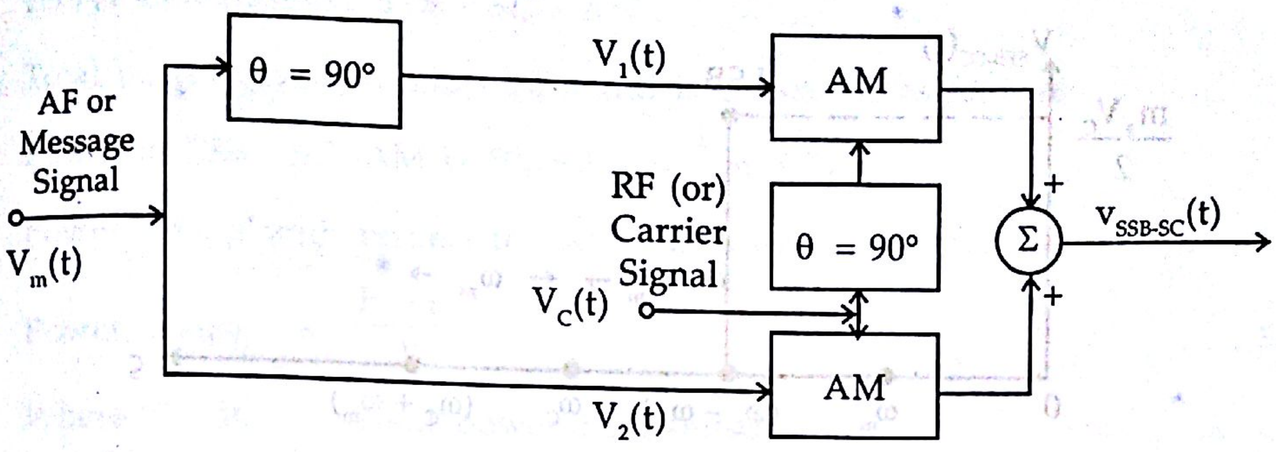


Figure 9 : Block diagram of SSB-SC-AM

The SSB - SC - AM can be obtained as follows.

In order to suppress one of the side bands, the input signal fed to the modulator 1 is 90° out of phase with that of the signal fed to the modulator '2'.

Let $V_1(t) = V_m \cdot \sin(\omega_m t + 90°)\, V_c \sin(\omega_C t + 90°)$ . . . 49

$$V_1(t) = V_m \cdot \cos\omega_m t.\ V_C \cos\omega_C t$$

$$V_2(t) = V_m \cdot \sin\omega_m t.\ V_C \sin\omega_C t \quad . . . 50$$

$$\therefore\ v(t)_{SSB} = V_1(t) + V_2(t) \quad . . . 51$$

$$= V_C V_m[\sin\omega_m t.\ \sin\omega_C t + \cos\omega_m t.\ \cos\omega_C t]$$

We know that $\sin A \sin B + \cos A \cos B = \frac{\cos(A - B)}{2}$

Hence $$\boxed{V(t)_{SSB} = \frac{V_m V_C}{2}\cos(\omega_C - \omega_m)t} \quad . . . 52$$

We know that for DSB-SC-AM

$$V_{DSB}(t) = \frac{V_m V_C}{2}[\cos(\omega_C - \omega_m)t - \cos(\omega_C + \omega_m)t] \quad . . . 53$$

When comparing equations 52 and 53 one of the side-band is suppressed. Hence this scheme is known as SSB-SC AM.

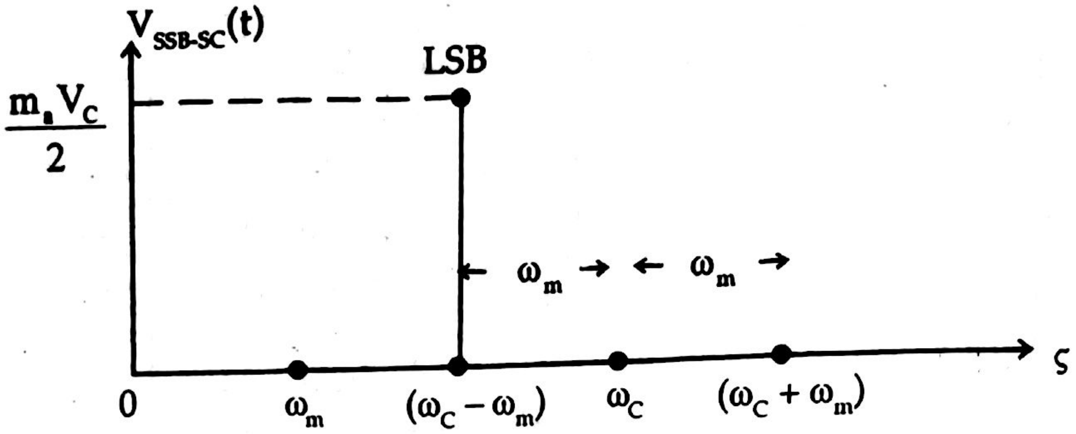


Figure 10 : Frequency spectrum of SSB - SC - AM

- The frequency spectrum of SSC - SC - AM is shown in figure 10. I shows that only one side band signal is present, the carrier and the other (upper) side band signal are suppressed. Thus the band width required reduces from $2\omega_m$ to ω_m. i.e., band width requirement is reduced to half compared to AM and DSB - SC signals.
- The graphical representation and phasor diagram of SSC - SC - AM system is shown in figure 11 and 12.

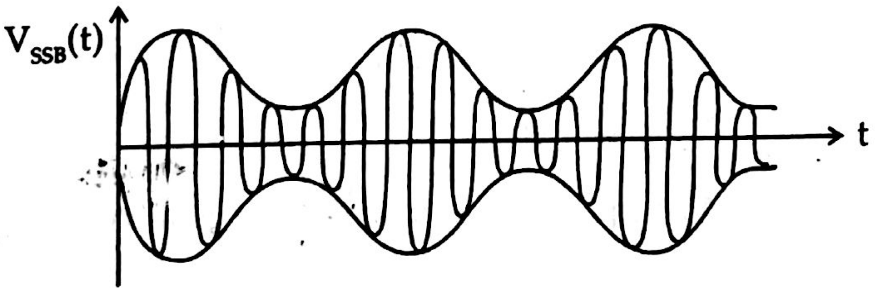


Figure 11 : Graphical representation of SSB-SC-AM

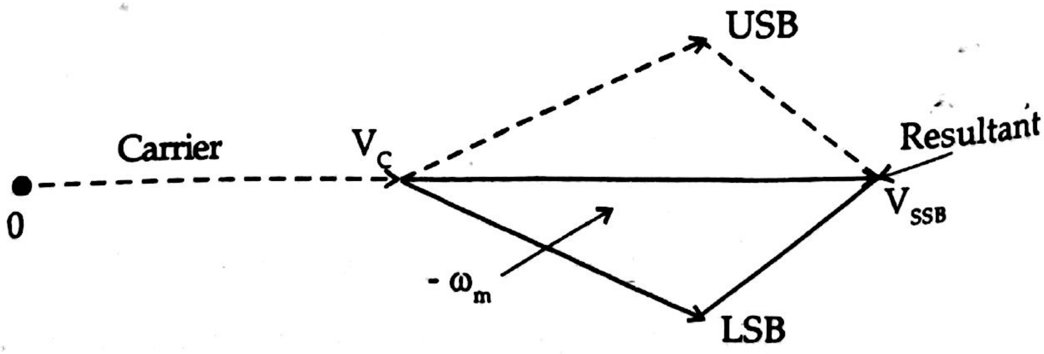


Figure 12 : Phasor diagram of SSB - SC - AM

Power Calculation : SSB - SC - AM :

Total power saved in SSB - SC - AM is calculated as follows

Power in SSB - SC -AM is $P''_t = P_{SB} = \frac{1}{4} m_a^2 P_C$

Power saving with respect to AM with carrier

$$\text{Power saving} = \frac{P_t - P_t''}{P_t} \qquad \ldots 54$$

Where P_t = Total power transmitted

$$\text{Power Saving} = \frac{\left[1+\frac{m_a^2}{2}\right]P_C - \left(\frac{m_a^2}{4}P_C\right)}{\left[1+\frac{m_a^2}{2}\right]P_C} = \frac{P_C + \frac{m_a^2 P_C}{2} - \frac{m_a^2 P_C}{4}}{\left[1+\frac{m_a^2}{2}\right]P_C}$$

$$= \frac{P_C + \frac{m_a^2 P_C}{4}}{\left[1+\frac{m_a^2}{2}\right]P_C} = \frac{\left[1+\frac{m_a^2}{4}\right]P_C}{\left[1+\frac{m_a^2}{2}\right]P_C} = \frac{\frac{\left(4+m_a^2\right)}{4}}{\frac{\left(2+m_a^2\right)}{2}} = \frac{4+m_a^2}{4+2m_a^2}$$

$$\text{If} \quad m_a = 1 \text{ then \% Power saving} = \frac{5}{6} = 83.3\ \% \qquad \ldots 55$$

Saving of power in SSB-SC AM with respect to AM with suppressed carrier (i.e., DSB - SC - AM).

$$= \frac{P_t' - P_t''}{P_t'} \qquad \ldots 56$$

$$= \frac{\frac{1}{2}m_a^2 P_C - \frac{1}{4}m_a^2 P_C}{\frac{1}{2}m_a^2 P_C} = \frac{\frac{1}{4}m_a^2 P_C}{\frac{1}{2}m_a^2 P_C} \qquad \ldots 57$$

$$= \frac{1}{2} \times 100 \qquad \ldots 58$$

% Power Saving = 50%

It has been concluded from the above analysis that in AM with carrier the total AM power is $\left[1+\frac{m_a^2}{2}\right]$ times the carrier power. If the carrier is suppressed, only the sidebands are transmitted, then 66.67% of power is saved. In addition to carrier, one of the sidebands is also suppressed, the power saving is 83.3% over AM with carrier.

Advantages and Disadvantages of SSB - SC - AM

Advantages

i) Band width of SSB - SC - AM is half that of DSB - SC AM. Thus twice the number of channels can be accommodated at a given frequency spectrum.

ii) No carrier is transmitted, hence possibility of interference with other channels are avoided.

iii) There is an improvement in signal to noise ratio from 9 to 12 db at the receiver output over DSB-SC-AM.

iv) During demodulation of SSB, carrier of same frequency and phase of requisite strength is to be inserted, and at the receiver one can get output audio signal without the knowledge of carrier. Hence some secrecy is automatically achieved.

v) It eliminates the possibility of fading. Fading occurs due to multipath propagation of electo-magnetic waves. Thus R.F. waves at same frequency may travel by two path which may be of different wave lengths so that signals received by these paths may be of unequal amplitude and phases, which results in fading. The fading is selective over the received band. If the transmitted signal consists of AM with carrier then the following three types of selective fading occurs.

   a) One side band fades completely leaving other sideband and carrier unaffected.

   b) Fading the carrier alone.

   c) Fading the amplitude and phase of one sideband component with respect to other side band and carrier.

Since in SSB only one side band is present, thus fading is eliminated.

vi) SSB provides an improvement in SNR of atleast 9db. Thus in order to get the same SNR at the receiver output the transmitter average power output may be reduced by 9db. Therefore SSB transmitter requires less number of amplifying stages. Hence net volume of operating cost is reduced.

Disadvantages

i) The major draw back is that the transmission and reception of SSB becomes more complex and the required performance standard is very high.

ii) For demodulation of SSB, carrier is reinserted at the receiver. The frequency of the reinserted carrier must be within 15 cycles per second of the carrier frequency in case of speech and 4 cycles per second in case of music. Such a requirement complicates the demodulation process. Hence it becomes necessary to transmit the pilot signal or the carrier voltage itself at a very low level for synchronising the receiver oscillator frequency. This signal has to be filtered out at the receiver with the use of highly selective filters. Design of these highly selective filters is thus involved in SSB receiver. This complexity contributes to an addition in cost.

Applications of SSB - SC - AM

Because of complexity and cost of SSB receiver this system is not used for commercial broadcasting. It is mainly used in

i) Police wireless communication.

ii) SSB telegraph system

iii) Point to point radio telephone communication

iv) VHF and UHF communication systems.

(iii)Instantaneous frequency (w_i)

It is the frequency of the carrier at any instant of time i.e.,

$$\omega_i(t) \quad = \quad \frac{d}{dt}\phi(t) \; = \; \frac{d}{dt}\,[\omega_C\, t + \theta(t)]$$

$$= \quad \omega_C + \theta'(t)$$

(iv) Instantaneous frequency deviation $\theta'(t)$

It is the change in frequency of the carrier. It can be defined as the fi time derivative of instantaneous phase derivation.

Deviation sensitivity

The deviation sensitivity provides relationship between output parame changes inrespect to input parameters for FM, the output frequency is vari inaccordance with the amplitude of the modulating signal.

i.e., $K_{FM} = \frac{\Delta\omega}{V_m} - \frac{\text{change in outpuf frequency}}{\text{change in input voltage}}$

similarly for PM, the output phase is varied w.r.to amplitude of modulati signal.

i.e., $K_{PM} = \frac{\Delta\theta}{V_m}$ or $\Delta\theta = K_{PM}\, V_M$

15. FREQUENCY MODULATION

"Frequency modulation" can be defined as the process by which the frequen of the carrier wave is altered in accordance with the instantaneous amplitu of modulating or message signal. The mathematical representation of frequen modulation is obtained as follows:

Let the message signal $v_m(t) = V_m \cos \omega_m t$ ...

and the carrier signal $v_C(t) = V_C \sin [\omega_C t + \theta]$ ...

Where V_m = maximum amplitude of message or modulati signal

V_C = maximum amplitude of carrier signal

ω_m = angular frequency of modulating signal

ω_C = angular frequency of carrier signal

ϕ = total instantaneous phase angle of carrier

ϕ = $(\omega_C t + \theta)$

$$\therefore \quad v_C(t) = V_C \sin \phi = V_C \sin (\omega_C t + \theta) \qquad \ldots 3$$

To find angular velocity, differentiate the equation (3) w.r.t. 't'

$$\text{i.e.,} \quad \frac{d\phi}{dt} = \omega_C = \phi'(t)$$

During the process of frequency modulation the frequency of carrier signal is changed in accordance with the instantaneous amplitude of message signal. Therefore the frequency of the carrier after modulation is written as

$$\omega_i = \omega_C + Kv_m(t) = \omega_C + KV_m \cos \omega_m t \qquad \ldots 4$$

Where K = Constant of proportionality.

To find the instantaneous phase angle of the modulated signal, integrate equation (4)

$$\phi_i = \int \omega_i \, dt = \int (\omega_C + KV_m \cos \omega_m t) dt = \omega_C t + \frac{KV_m}{\omega_m} \sin \omega_m t + \theta_1$$

θ_1 = Integration constant, it is neglected because it plays no role in modulation process.

The instantaneous amplitude of the modulating signal is given by

$$v(t)_{FM} = V_C \sin \phi_1 = V_C \sin (\omega_C t + \frac{KV_m}{\omega_m} \sin \omega_m t) \qquad \ldots. 5$$

$$\boxed{v(t)_{FM} = V_C \sin (\omega_C t + m_f \sin \omega_m t)} \qquad \ldots. 6$$

Where $\boxed{m_f = \frac{K_{FM} V_m}{\omega_m} \text{ modulation index of FM}}$

From equation (4) the instantaneous angular frequency of FM signal is $\omega_i = \omega_C + KV_m \cos \omega_m t$

The maximum and minimum value of cosine term is ±1. Hence the maximum value of angular frequency is given by $\omega_{max} = \omega_C + KV_m$

The minimum value of angular frequency is given by $\omega_{min} = \omega_C - KV_m$

Then **frequency deviation** is given by

$$\boxed{\omega_d = \omega_{max} - \omega_C = \omega_C - \omega_{min} = KV_m} \quad \ldots 7$$

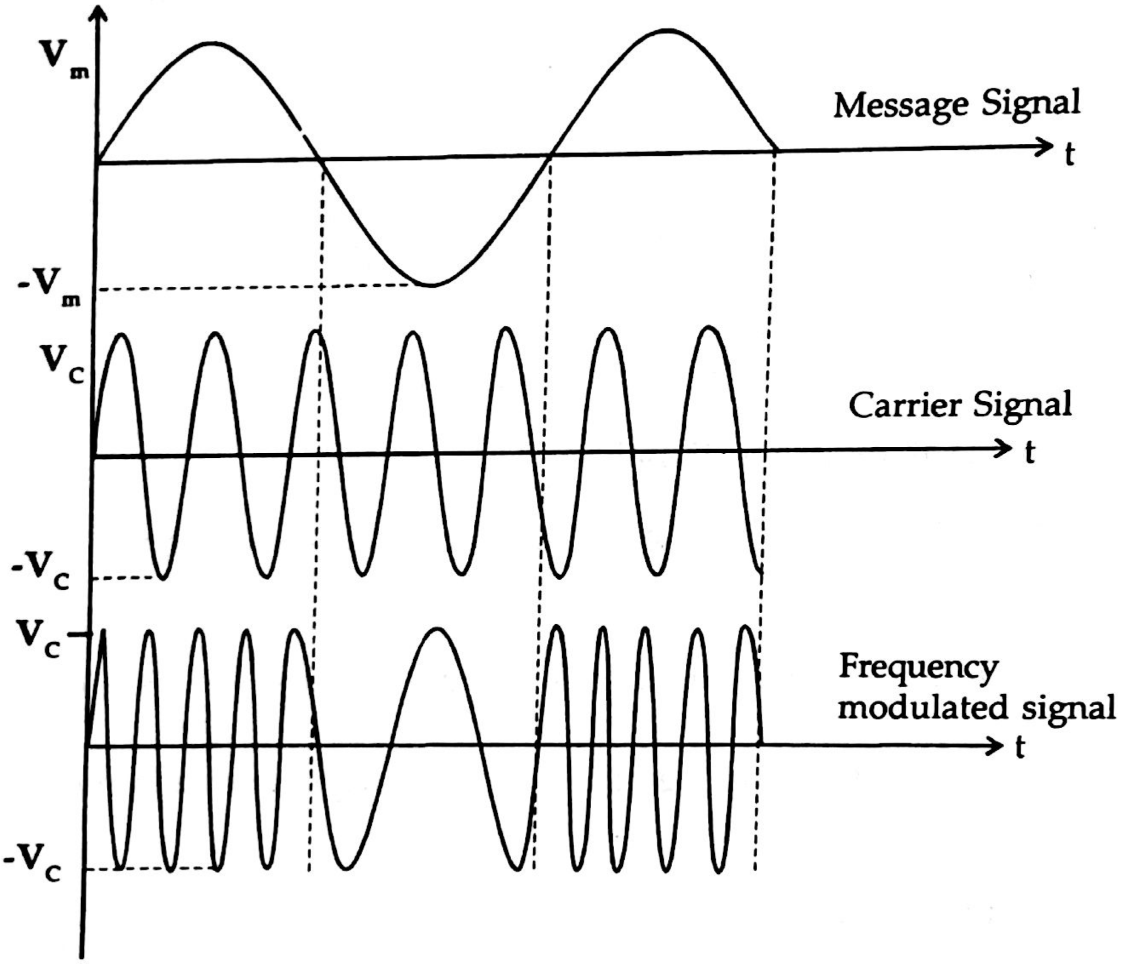


Figure 17 : Graphical representation of FM wave

16. PHASE MODULATION

Phase modulation can be defined as the process by which changing the phase of the carrier signal in accordance with the instantaneous amplitude of the message signal. The amplitude and frequency remains constant even after the modulation process.

Let the modulating signal is given by $v_m(t) = V_m \cos \omega_m t$

The carrier signal $v_C(t) = V_C \sin (\omega_C t + \theta)$

where θ = phase angle of carrier signal. It is changed in accordance with the amplitude of the message signal $v_m(t)$;

i.e., $\theta = K_{PM}v_m(t) = K_{PM}V_m\cos\omega_m t$ . . . 8

where K_{PM} = phase deviation sensitivity

After phase modulation the instantaneous voltage will be

$$v_{pm}(t) = V_C \sin(\omega_C t + \theta)$$

$$= V_C \sin(\omega_C t + KV_m \cos\omega_m t) \quad \ldots 9$$

$$\boxed{v_{pm}(t) = V_C \sin(\omega_C t + m_p \cos\omega_m t)} \quad \ldots 10$$

where $m_p = KV_M$ Modulation index of phase modulation

17. PHASE DEVIATION AND MODULATION INDEX

The equation (6) compared with equation (3) thus we get

$$v_{fm}(t) = V_C \sin(\omega_C t + m_f \sin\omega_m t) = V_C \sin[\omega_C t + \theta(t)]$$

where $\theta(t)$ = instantaneous phase deviation = $m_f \sin\omega_m t$.

If the modulating signal is single tone or sinusoid, then the phase angle of the carrier varies from its unmodulated signal is known as **phase deviation.**

The "**Modulation Index**" of FM System can be defined as the ratio of maximum frequency deviation to the modulating frequency.

i.e. $m_f = \frac{\omega_d}{\omega_m} = \frac{K_{PM}V_m}{\omega_m} = \delta$ . . . 8

$\omega_d = KV_m$ = Maximum frequency deviation

For PM

The modulation index depends on the modulating signal

i.e., $\boxed{m_p = K_{PM} V_m}$ where K_{PM} = deviation sensitivity.

Frequency deviation

It is defined as the change in frequency of the carrier with respect to amplitude of the modulating signal, it can be written as $\Delta\omega = K_{fm}.V_m$ where K_{fm} = deviation sensitivity, interms of modulation index, it can be written

as $m = \frac{\Delta\omega}{\omega_m} = \frac{\Delta f}{f_m}$ (or) $\Delta f = m_f f_m$

2. LOW PASS SAMPLING THEOREM

The sampling process is an operation that is basic to digital signal processing and digital communication. Through the **use of sampling process** an analog signal is converted into corresponding sequences of sample pulses that are equally spaced in time. It is necessary to choose sampling rate properly so that the sequence of samples uniquely defines the original analog signal.

"**Sampling**" is the process by which an analog signal is converted into a corresponding sequence of samples that are spaced uniformly in time (i.e., equally spaced in time). In this process, it is necessary to choose the sampling rate properly, so that the sequence of samples uniquely defines or recovers the original analog signal. This is the essence of sampling theorem.

The sampling theorem states that, any band limited signal (i.e., low pass filtered signal) which has no spectral components above the frequency f_m Hz is uniquely determined by its values at uniform intervals less than $1/2f_m$ seconds apart.

Consider an arbitrary analog signal f(t) of finite energy, as shown in figure 2. It is sampled instantaneously at a uniform rate, once in every T_s seconds. Consequently we obtain an infinite sequence of samples spaced T_s seconds apart and denoted by $f(nT_S)$.

Where n ⇒ takes on all possible integer values or discrete time in sec.

T_S ⇒ Sampling period in sec.

The reciprocal of sampling period is called the "**sampling frequency**" or "**sampling, rate**" i.e. $f_S = \frac{1}{T_s}$. This ideal form of sampling is called "**Instantaneous sampling**". If the signal is sampled at an equal or uniform intervals then it is known as "**Uniform sampling**".

The sampled function $f_s(t)$ may be written as

$$f_s(t) = f(t)^*G(t) \qquad (1)$$

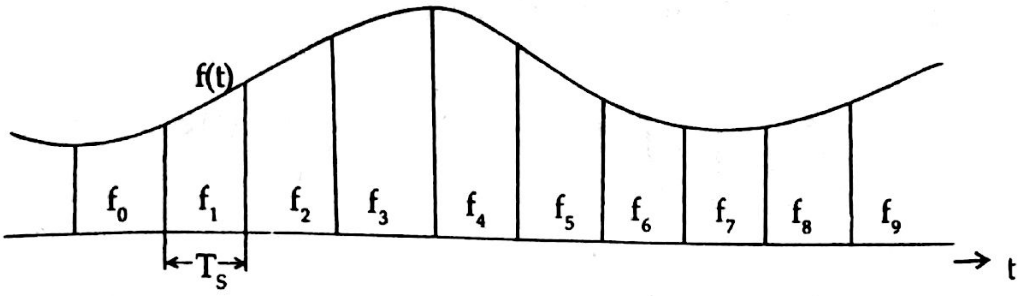
where $G(t)$ = train of impulses = $\sum_{n=-\infty}^{\infty} \delta(t - nT)$

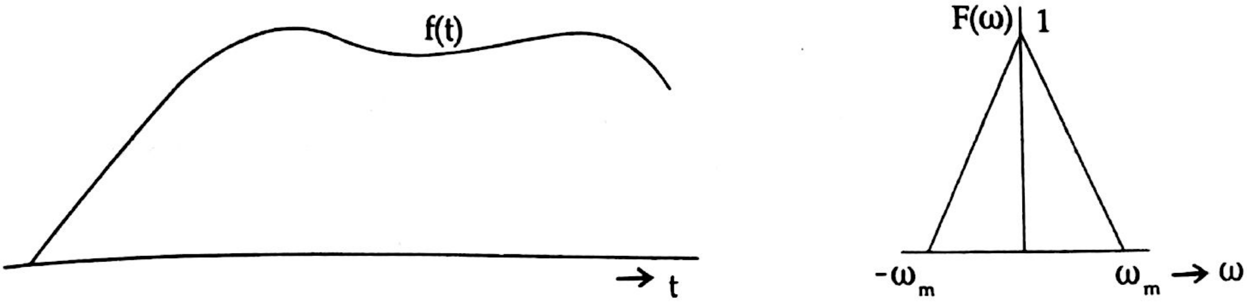
thus equation (1) becomes

$$f_s(t) = f(t) * \sum_{n=-\infty}^{\infty} \delta(t - nT_s) = \sum_{n=-\infty}^{\infty} f(t) * \delta(t - nT_s) \qquad (2)$$

The G(t) is a periodic impulse train it may be represented using Fourier series representation

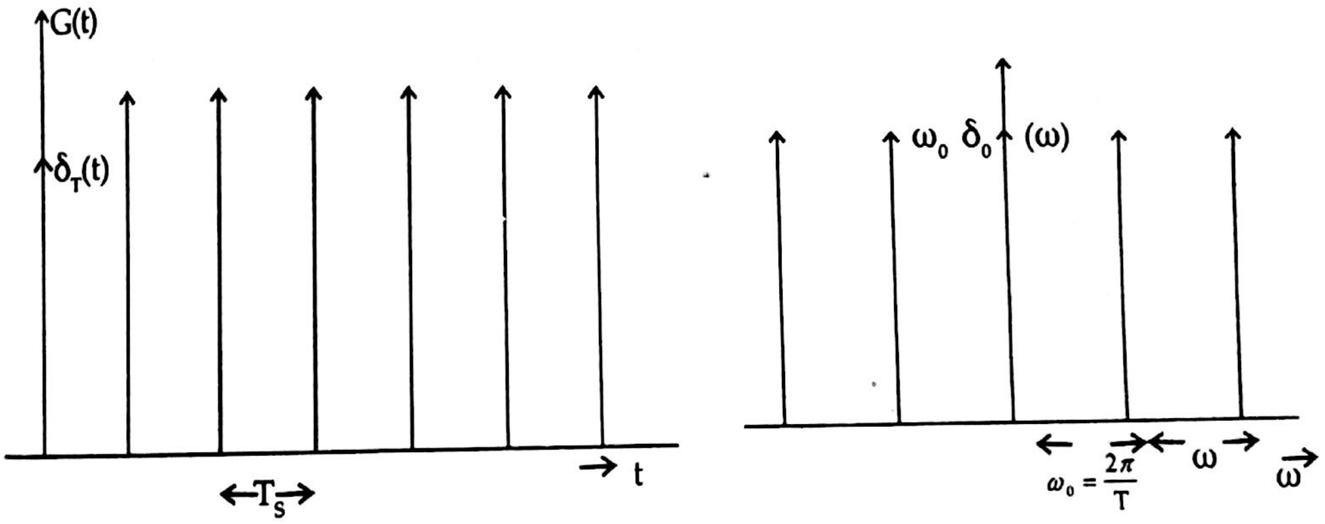
$$G(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_s t} \text{ (in exponential form)}$$





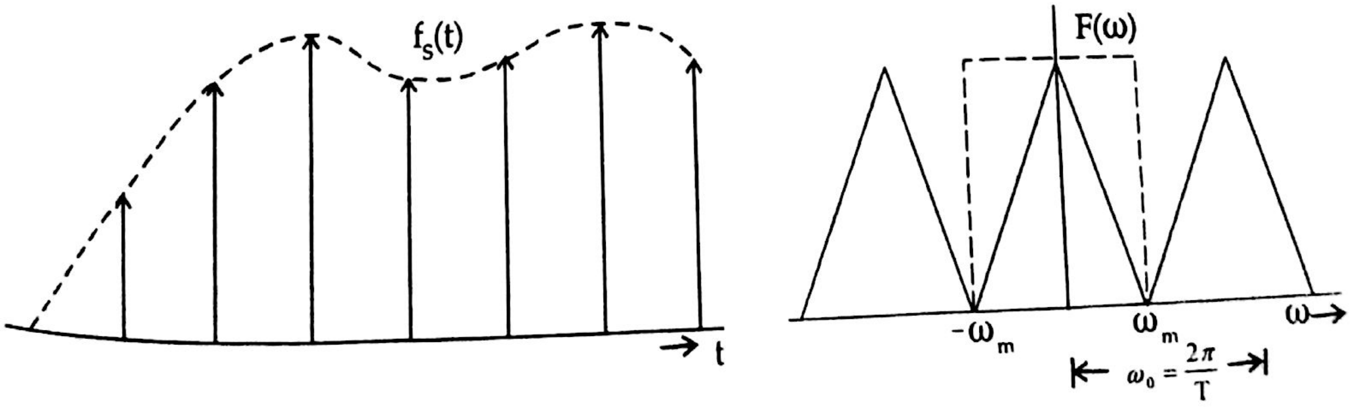
(a) Modulating signal

(b) Frequency spectrum for modulating signal



(c) Train of Impulse carrier signal

(d) Frequency spectrum for Impulse signal



(e) Sampled signal

(f) Frequency spectrum for sampled signal

Figure 2 : Sampling theorem

where F_n = Fourier coefficient

$$= \frac{1}{T}\int_{-T/2}^{T/2} G(t)\, e^{-jn\omega_s t}\, dt = \frac{1}{T}\int_{o}^{T} G(t)\, e^{-jn\omega_s t\, dt}$$

thus
$$F_n = \frac{1}{T}\int_{-T/2}^{T/2} \delta(t - nT_s)\, e^{-n\omega_s t}\, dt = \frac{1}{T} \qquad (3)$$

so
$$G(t) = \sum_{n=-\infty}^{\infty} \left(\frac{1}{T}\right) e^{jn\omega_s t}$$
substitute this value in equation (2)

now
$$f_s(t) = \frac{1}{T}\sum_{n=-\infty}^{\infty} f(t) * e^{jn\omega_s t} \qquad (4)$$

Convolution in time domain is equal to multiplication in frequency domain, thus equation (4) is converted into frequency domain.

Take F.T on bothsides of equation (4) we get

$$\text{F.T}\,[f_s(t)] = F_s(\omega)$$

$$\text{F.T}\,[f(t)] = F(\omega)$$

$$\text{F.T}\,[e^{jn\omega st}] = 2\pi\ \delta(\omega - n\omega_s)$$

thus equation (4) becomes

$$F_s(\omega) = \frac{1}{T}\sum_{n=-\infty}^{\infty} F(\omega).2\pi\, \delta(\omega - n\omega_s)$$

$$= \frac{2\pi}{T}\sum_{n=-\infty}^{\infty} F(\omega)\, \delta\,(\omega - n\omega_s) \qquad (5)$$

$$= \omega_s \sum_{n=-\infty}^{\infty} F(\omega)\, \delta\,(\omega - n\omega_s)$$

$$= \omega_s \sum_{n=-\infty}^{\infty} F\,(\omega - n\omega_s) \qquad (6)$$

Now the aliasing error becomes

$$\varepsilon \quad = \quad |f(t) - f_i(t)| = \left| \sum_{m=-\infty}^{\infty} (1 - e^{-jm\omega_s t}) \int_{(m-1/2)\omega_s}^{(m+1/2)\omega_s} F(\omega)\, e^{j\omega t}\, d\omega \right| \qquad (6)$$

if m = 0, then $1 - e^{jm\omega_s t} \leq 2$ and

$$\left| \int_{(m-1/2)\omega_s}^{(m+1/2)\omega_s} F(\omega)\, e^{j\omega t}\, d\omega \right| \leq \int_{(m-1/2)\omega_s}^{(m+1/2)\omega_s} |F(\omega)|\, d\omega \qquad (7)$$

thus

$$\varepsilon \leq 2 \int_{|\omega| > \omega_s/2} |F(\omega)|\, d\omega. \qquad (8)$$

4 (4. PRACTICAL ASPECTS OF SAMPLING / SAMPLING TECHNIQUES

In practice, there are three types of sampling techniques are used to convert continuous time signal into discrete time signal (i) Instantaneous sampling, (ii) Natural sampling (iii) Flattop sampling.

(i) Instantaneous sampling

The proof of sampling theorem is the example for instantaneous or Impulse sampling.

(ii) Natural sampling

Let us consider a unit impulse train, each pulse is separated by T_s. The sampled sequence is obtained by multiplying f(t) with train of impulses hence the resultant signal is obtained as shown in figure 5. It may be seen from the top, the pulses are not flat, but they follow the natural waveform of input signal f(t) during respective pulse intervals and hence it is named as "**natural sampling**".

The Fourier series representation of carrier pulses can be written as

$$v(t) \quad = \quad \frac{A\tau}{T_0} + \frac{2A\tau}{T_0} \sum_{n=-\alpha}^{\alpha} C_n \mathrm{Cos}\left[\frac{2\pi nt}{T_0}\right] \qquad (1)$$

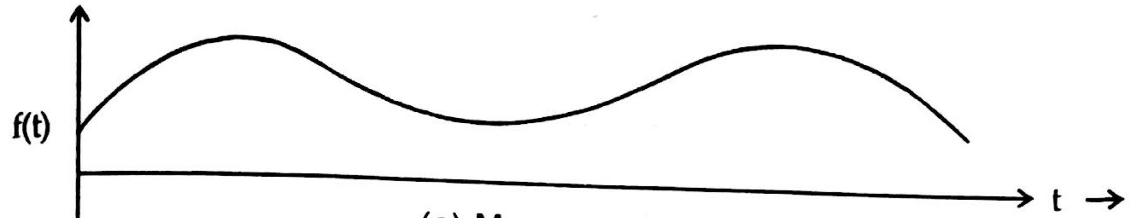
where A = amplitude of the pulse

τ = duration of the pulse

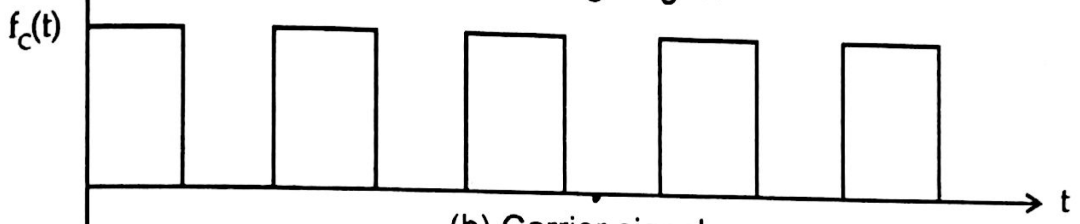
T_o = period of the pulse train

$$C_n = \frac{\text{Sin}\,(n\pi\tau\ /\ T_o)}{(n\pi\tau\ /\ T_o)} \quad \text{Assume } A = 1,\ T_0 = T_s = \frac{1}{2f_m}$$

then $$f_c(t) = v(t) = \frac{\tau}{T_s} + \frac{2\tau}{T_s}\left[C_1\ \text{Cos}\left(\frac{2\pi t}{T_s}\right) + C_2\ \text{Cos}\left(\frac{4\pi t}{T_s}\right) + \ldots\right] \qquad (2)$$



(a) Message signal



(b) Carrier signal

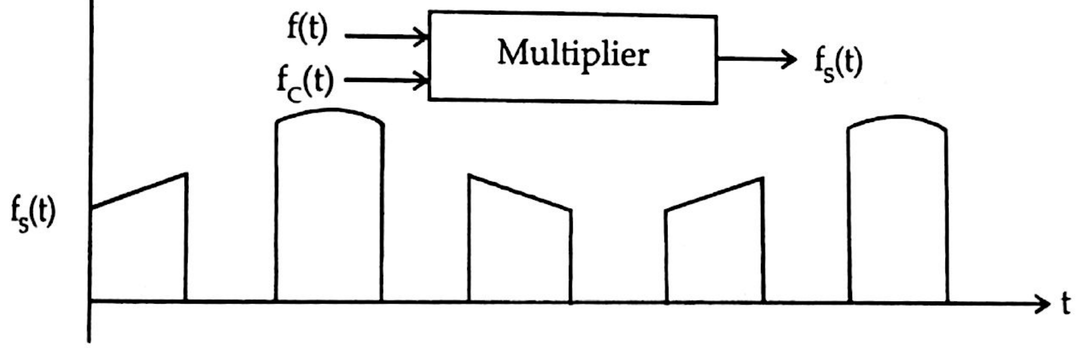


Figure 4 : Natural sampling

The carrier pulses are multiplied by the message signal thus the output of the multiplier is given by (Assume $f(t) = A\ \text{Cos}\omega_m t$)

$$f_s(t) = f(t).f_c(t) = \frac{\tau}{T_s}\ f(t) + \frac{2\tau}{T_s}\left[f(t)\ C_1\,\text{Cos}\left(\frac{2\pi t}{T_s}\right) + f(t)\ C_2\ \text{Cos}\left(\frac{4\pi t}{T_s}\right) + \ldots\right]$$

$$= \frac{\tau}{T_s}\ f(t) + \frac{2\tau}{T_s}\left[f(t)\ C_1\ \text{Cos}2\pi(2f_m)t\ +\ f(t)\ C_2\ \text{Cos}2\pi(4f_m)t\ +\ \ldots\ \right]$$

$$= \frac{\tau}{T_s}\ A\text{cos}\omega_m t + \frac{2\tau}{T_s}[C_1 A\ \text{Cos}\ \omega_m t\ .\ \text{Cos}\ 2\pi(2f_m)t + \ldots..]$$

$$= \frac{\tau}{T_s} A\cos\omega_m t + \frac{2\tau}{T_s}[C_1 A \{\cos(2\omega_m + \omega_m)t + \cos(2\omega_m - \omega_m t)\} + \ldots.. + C_n A\{\cos(2\omega_m + n\omega_m)t + \cos(2\omega_m - n\omega_m)t\} \qquad (3)$$

(iii) Flat-top Sampling

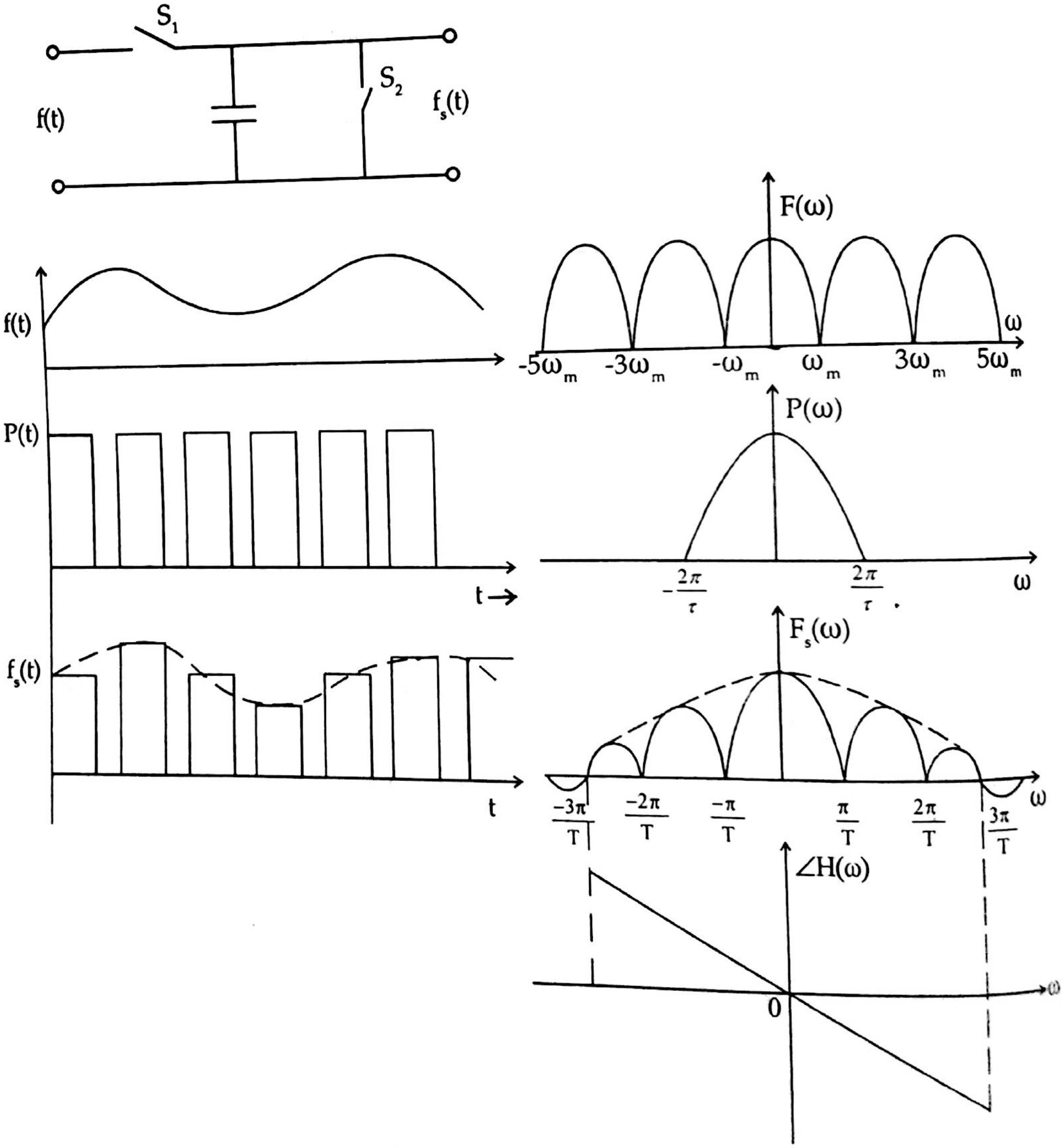


Figure 5 : Flat top sampling

The electronics circuitry needed to perform natural sampling is complicated because the pulse top shape is to be maintained. These complications are reduced by flat top sampling. In this method the tops of the pulses are flat. Thus the pulses have a constant amplitude within the pulse interval. The constant amplitude of pulse can be chosen at any value of f(t) within the pulse interval. By making the value of the pulse amplitude constant within the pulse interval, some distortion is introduced as there is a deviation from the actual value of f(t).

Figure 5 shows the circuit and the mathematical model for generating a flat top sampled signal. The switch S_1 closes at each sampling instant in order to sample m(t). The capacitor C holds the sampled voltage for a time period τ at the end of which S_2if closed to fully discharge the capacitor. Thus a flat top sampled signal is generated by a process of sample and hold.

The flat top sampled signal may be considered as a convolution of the impulse sampled signal $f_s(t)$ and non periodic pulse p(t) of width 'τ' and height '1'. The spectrums of $F_s(t)$ and P(t) are shown in figure 5. The spectrum of $f_s(t)$ is obtained by multiplying F(t) with P(t). As the $P(\omega)$ value is different at different frequencies, the shape of $F_s(\omega)$ is not similar to $F(\omega)$ which shows that a distortion will be introduced if the signal is recovered by an ideal lowpass filter of a cutoff frequency ω_m.

2. *Define TDM.* (2 Marks)

2.5 Pulse Code Modulation

2.5.1 PCM Generator

The pulse code modulator technique samples the input signal $x(t)$ at frequency $f_s \geq 2W$. This sampled 'Variable amplitude' pulse is then digitized by the analog to digital converter. The parallel bits obtained are converted to a serial bit stream. Fig.2.5.1 shows the PCM generator.

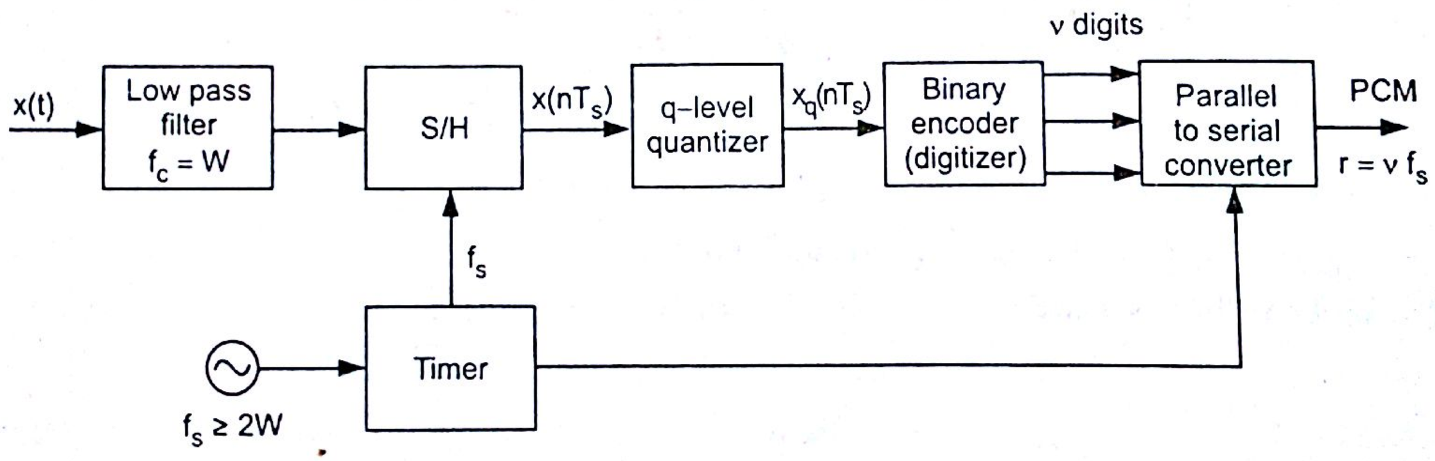


Fig. 2.5.1 PCM generator

In the PCM generator of above figure, the signal $x(t)$ is first passed through the low-pass filter of cutoff frequency 'W' Hz. This low-pass filter blocks all the frequency components above 'W' Hz. Thus $x(t)$ is bandlimited to 'W' Hz. The sample and hold circuit then samples this signal at the rate of f_s. Sampling frequency f_s is selected sufficiently above Nyquist rate to avoid aliasing i.e.,

$$f_s \geq 2W$$

In Fig. 2.5.1 output of sample and hold is called $x(nT_s)$. This $x(nT_s)$ is discrete in time and continuous in amplitude. A q-level quantizer compares input $x(nT_s)$ with its fixed digital levels. It assigns any one of the digital level to $x(nT_s)$ with its fixed digital levels. It then assigns any one of the digital level to $x(nT_s)$ which results in minimum distortion or error. This error is called *quantization error*. Thus output of quantizer is a digital level called $x_q(nT_s)$.

Now coming back to our discussion of PCM generation, the quantized signal level $x_q(nT_s)$ is given to binary encoder. This encoder converts input signal to 'v' digits binary word. Thus $x_q(nT_s)$ is converted to 'V' binary bits. The encoder is also called digitizer.

It is not possible to transmit each bit of the binary word separately on transmission line. Therefore 'v' binary digits are converted to serial bit stream to generate single baseband signal. In a parallel to serial converter, normally a shift register does this job. The output of PCM generator is thus a single baseband signal of binary bits.

An oscillator generates the clocks for sample and hold an parallel to serial converter. In the pulse code modulation generator discussed above ; sample and hold, quantizer and encoder combinely form an analog to digital converter.

2.5.2 Transmission Bandwidth in PCM

Let the quantizer use 'v' number of binary digits to represent each level. Then the number of levels that can be represented by 'v' digits will be,

$$q = 2^v \qquad \text{... (2.5.1)}$$

Here 'q' represents total number of digital levels of q-level quantizer.

For example if $v = 3$ bits, then total number of levels will be,

$$q = 2^3 = 8 \text{ levels}$$

Each sample is converted to 'v' binary bits. i.e. Number of bits per sample = v

We know that, Number of samples per second = f_s

∴ Number of bits per second is given by,

$$\begin{aligned}(\text{Number of bits per second}) &= (\text{Number of bits per samples}) \\ &\quad \times (\text{Number of samples per second}) \\ &= v \text{ bits per sample} \times f_s \text{ samples per second} \qquad \text{... (2.5.2)}\end{aligned}$$

The number of bits per second is also called signaling rate of PCM and is denoted by 'r' i.e.,

$$\boxed{\textbf{Signaling rate in PCM : } r = v f_s}$$

Here $f_s \geq 2W$. ... (2.5.3)

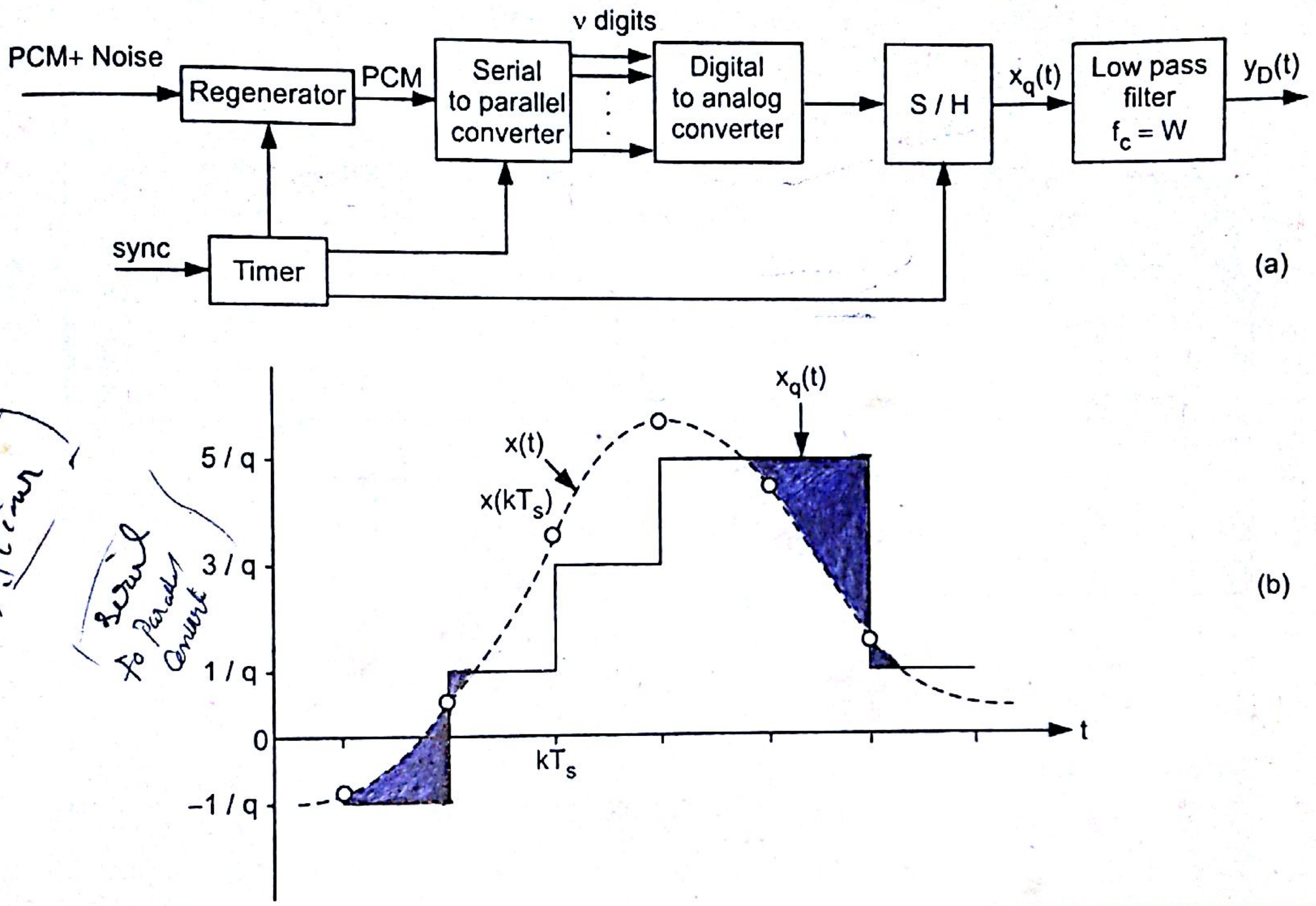
Bandwidth needed for PCM transmission will be given by half of the signaling rate i.e.,

Transmission Bandwidth of PCM :

$$\left\{\begin{aligned} &B_T \geq \frac{1}{2} r && \text{... (2.5.4)} \\ &B_T \geq \frac{1}{2} v f_s \quad \text{Since } f_s \geq 2W && \text{... (2.5.5)} \\ &B_T \geq v W && \text{... (2.5.6)} \end{aligned}\right.$$

2.5.3 PCM Receiver

Fig. 2.5.2 (a) shows the block diagram of PCM receiver and Fig. 2.5.2 (b) shows the reconstructed signal. The regenerator at the start of PCM receiver reshapes the pulses and removes the noise. This signal is then converted to parallel digital words for each sample.



Timer

Serial to Parallel Convert

Fig. 2.5.2 (a) PCM receiver
(b) Reconstructed waveform

The digital word is converted to its analog value $x_q(t)$ along with sample and hold. This signal, at the output of S/H is passed through lowpass reconstruction filter to get $y_D(t)$. As shown in reconstructed signal of Fig. 2.5.2 (b), it is impossible to reconstruct exact original signal $x(t)$ because of permanent quantization error introduced during quantization at the transmitter. This quantization error can be reduced by increasing the binary levels. This is equivalent to increasing binary digits (bits) per sample. But increasing bits 'v' increases the signaling rate as well as transmission bandwidth as we have seen in equation 2.5.3 and equation 2.5.6. Therefore the choice of these parameters is made, such that noise due to quantization error (called as quantization noise) is in tolerable limits.

2.5.4 Uniform Quantization (Linear Quantization)

We know that input sample value is quantized to nearest digital level. This quantization can be uniform or nonuniform. In uniform quantization, the quantization step or difference between two quantization levels remains constant over the complete *and thermal noise.*

2.6 Delta Modulation

We have seen in PCM that, it transmits all the bits which are used to code the sample. Hence signaling rate and transmission channel bandwidth are large in PCM. To overcome this problem Delta Modulation is used.

Delta modulation transmits only one bit per sample. That is the present sample value is compared with the previous sample value and the indication,whether the amplitude is increased or decreased is sent. Input signal $x(t)$ is approximated to step signal by the delta modulator. This step size is fixed . The difference between the input signal $x(t)$ and staircase approximated signal confined to two levels, i.e.

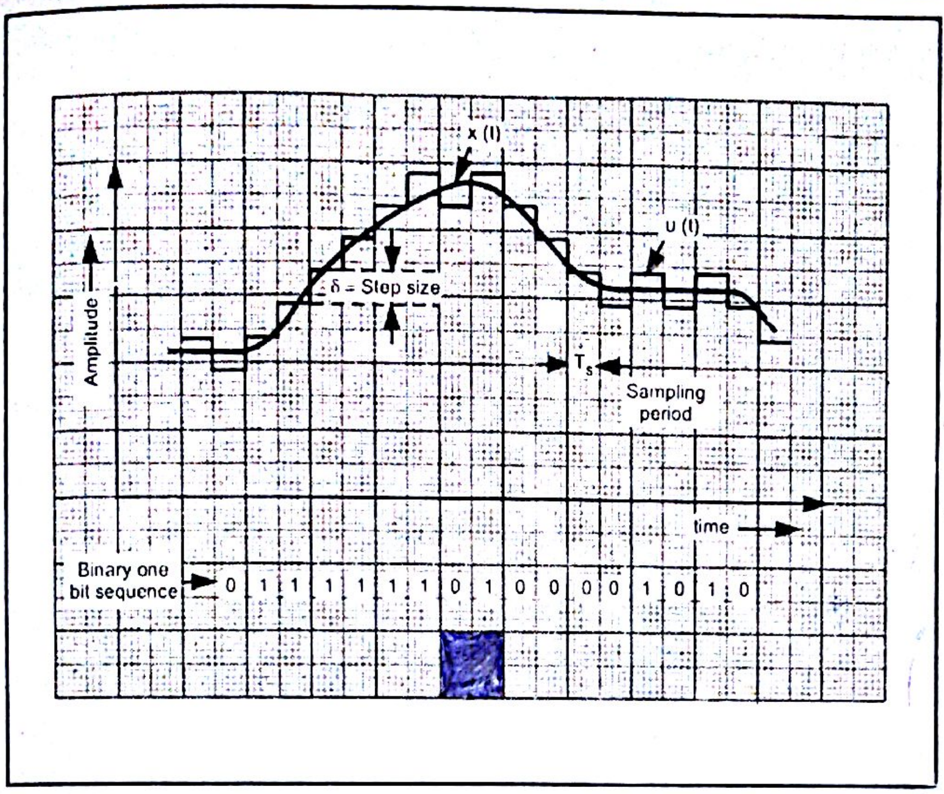


Fig. 2.6.1 Delta modulation waveform

+δ and −δ. If the difference is positive, then approximated signal is increased by one step i.e. 'δ'. If the difference is negative, then approximated signal is reduced by 'δ'. When the step is reduced, '0' is transmitted and if the step is increased, '1' is transmitted. Thus for each sample, only one binary bit is transmitted. Fig. 2.6.1 shows the analog signal $x(t)$ and its staircase approximated signal by the delta modulator.

The principle of delta modulation can be explained by the following set of equations. The error between the sampled value of $x(t)$ and last approximated sample is given as,

$$e(nT_s) = x(nT_s) - \hat{x}(nT_s) \qquad \ldots (2.6.1)$$

Here, $e(nT_s)$ = error at present sample

1 - 6

$x(nT_s)$ = Sampled signal of $x(t)$

$\hat{x}(nT_s)$ = Last sample approximation of the staircase waveform.

We can call $u(nT_s)$ as the present sample approximation of staircase output.

Then, $u[(n-1)T_s] = \hat{x}(nT_s) \qquad \ldots (2.6.2)$

= Last sample approximation of staircase waveform.

Let the quantity $b(nT_s)$ be defined as,

$$b(nT_s) = \delta \, sgn[e(nT_s)] \qquad \ldots (2.6.3)$$

That is depending on the sign of error $e(nT_s)$ the sign of step size δ will be decided. In other words,

$$\begin{aligned} b(nT_s) &= +\delta && \text{if} \quad x(nT_s) \geq \hat{x}(nT_s) \\ &= -\delta && \text{if} \quad x(nT_s) < \hat{x}(nT_s) \qquad \ldots (2.6.4) \end{aligned}$$

If $b(nT_s) = +\delta$; binary '1' is transmitted

and if $b(nT_s) = -\delta$; binary '0' is transmitted.

T_s = Sampling interval.

Fig. 2.6.2 (a) shows the transmitter based on equations 2.6.3 to 2.6.5.

The summer in the accumulator adds quantizer output ($\pm\delta$) with the previo sample approximation. This gives present sample approximation. i.e.,

$$u(nT_s) = u(nT_s - T_s) + [\pm\delta] \qquad \text{or}$$
$$= u[(n-1)T_s] + b(nT_s) \qquad \ldots (2.6$$

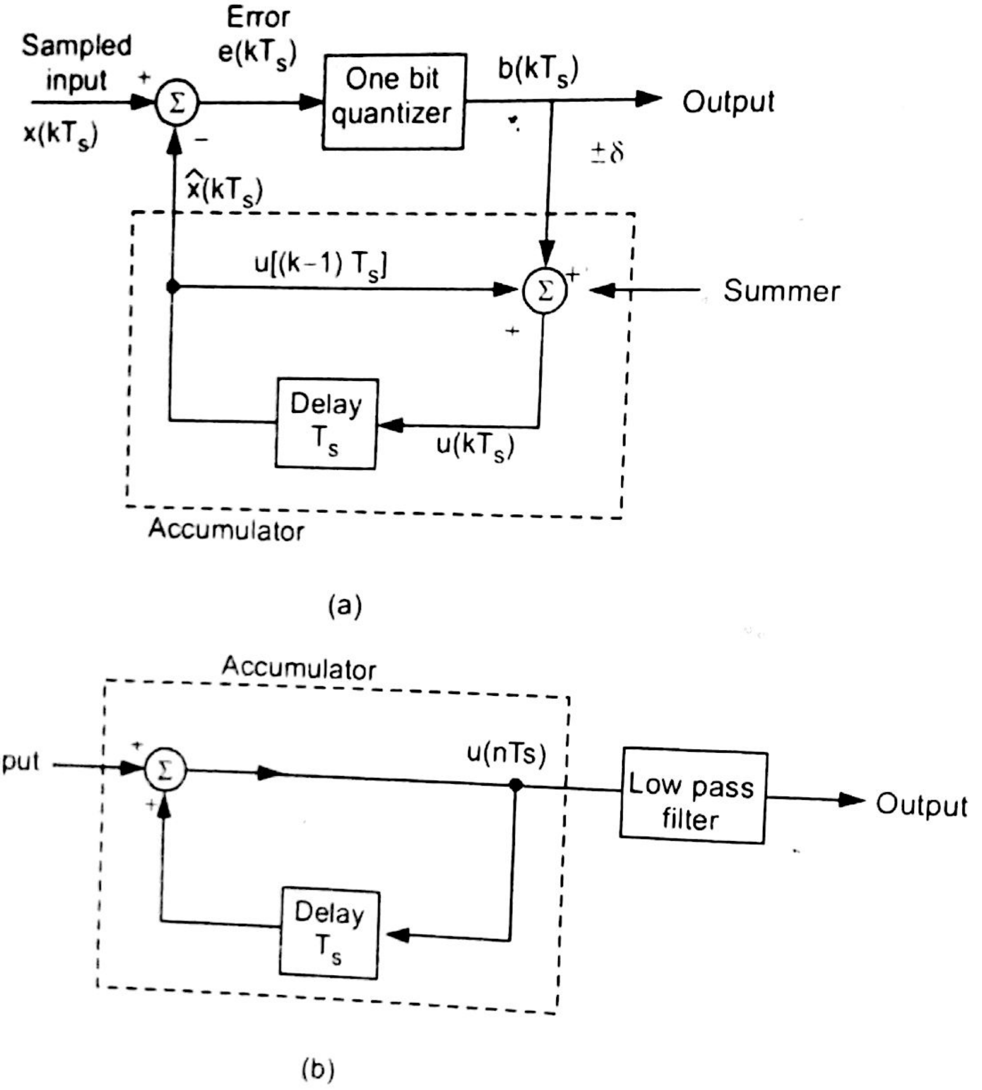


Fig. 2.6.2 (a) Delta modulation transmitter and (b) Delta modulation receiver

The previous sample approximation $u[(n-1)T_s]$ is restored by delaying one sample period T_s. The sampled input signal $x(nT_s)$ and staircase approximated signal $\hat{x}(nT_s)$ are subtracted to get error signal $e(nT_s)$.

Depending on the sign of $e(nT_s)$ one bit quantizer produces an output step of $+\delta$ or $-\delta$. If the step size is $+\delta$, then binary '1' is transmitted and if it is $-\delta$, then binary '0' is transmitted.

At the receiver shown in Fig. 2.6.2 (b), the accumulator and low-pass filter are used. The accumulator generates the staircase approximated signal output and is delayed by one sampling period T_s. It is then added to the input signal. If input is binary '1' then it adds $+\delta$ step to the previous output (which is delayed). If input is binary '0' then one step 'δ' is subtracted from the delayed signal. The low-pass filter has the cutoff frequency equal to highest frequency in $x(t)$. This filter smoothen the staircase signal to reconstruct $x(t)$.

2.6.1 Advantages of Delta Modulation

$u(nT_s] = u[(n-1)T_s + \delta$

The delta modulation has following advantages over PCM,

1. Delta modulation transmits only one bit for one sample. Thus the signaling rate and transmission channel bandwidth is quite small for delta modulation.
2. The transmitter and receiver implementation is very much simple for delta modulation. There is no analog to digital converter involved in delta modulation.

2.6.2 Disadvantages of Delta Modulation

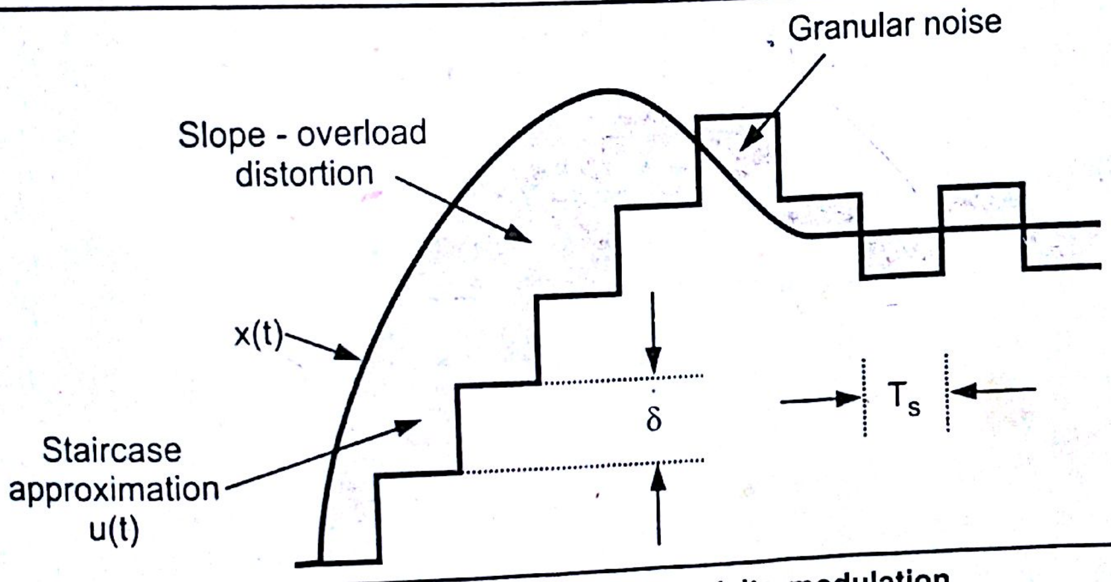


Fig. 2.6.3 Quantization errors in delta modulation

The delta modulation has two drawbacks -

(Startup Error)

$\left(\right)_{DPCM}$ $\left(V_{rms}\right)$ $\left(\right)$

13. COMPARISON OF DIGITAL PULSE MODULATION

	Parameter	PCM	DM	ADM	DPCM
1.	Number of bits per sample	4 or 8 or 16 bits used per sample	Only one bit per sample	Only one bits per sample	More than 1 but less than PCM
2.	Step size	depends on the number of bits	Fixed can not be varied	Adaptive i.e., variable depending upon the signal variation	Fixed
3.	Bandwidth	High	Low	Low	Less than PCM
4.	Generation	Complex	Simple	Simple	Simple
5.	Quantization noise or distortion	depends on number of levels used	slope over load and granular noise occurs	only quantization error occurs	both quantization and slope overload noise occurs
6.	SNR	Good	Poor	Better	12db greater than PCM
7.	Sampling rate	8KHz	64-128KHz	48 - 64 KHz	8KHz
8.	Bit rate	7-8, thus high bit rate PCM is superior	1, so it is suitable for low bit rate	1 suitable for low bit rate	4-6
9.	Application	Telephony	Audio and speech processing	Audio and speech processing	Audio and speech Processing.

Alternate method : Mathematical proof for ISI

26 In discrete pulse modulation, the amplitude, time or frequency of the transmitted pulse is varied according to the information to be transmitted. In this method PAM systems are most efficient than other systems, interms of power and bandwidth utilization. The elements of base band binary PAM system are shown in figure 30.

Actually the base band binary data transmission means binary data transmitted over a coaxial cable. The input to the system is a binary sequences with a bit rate of r_b and bit duration of 'T_b'. The pulse generator output is a pulse waveform. It can be written as,

a_K = a if the 'K'th input bit is '1'

= – a if the 'K'th input bit is '0'

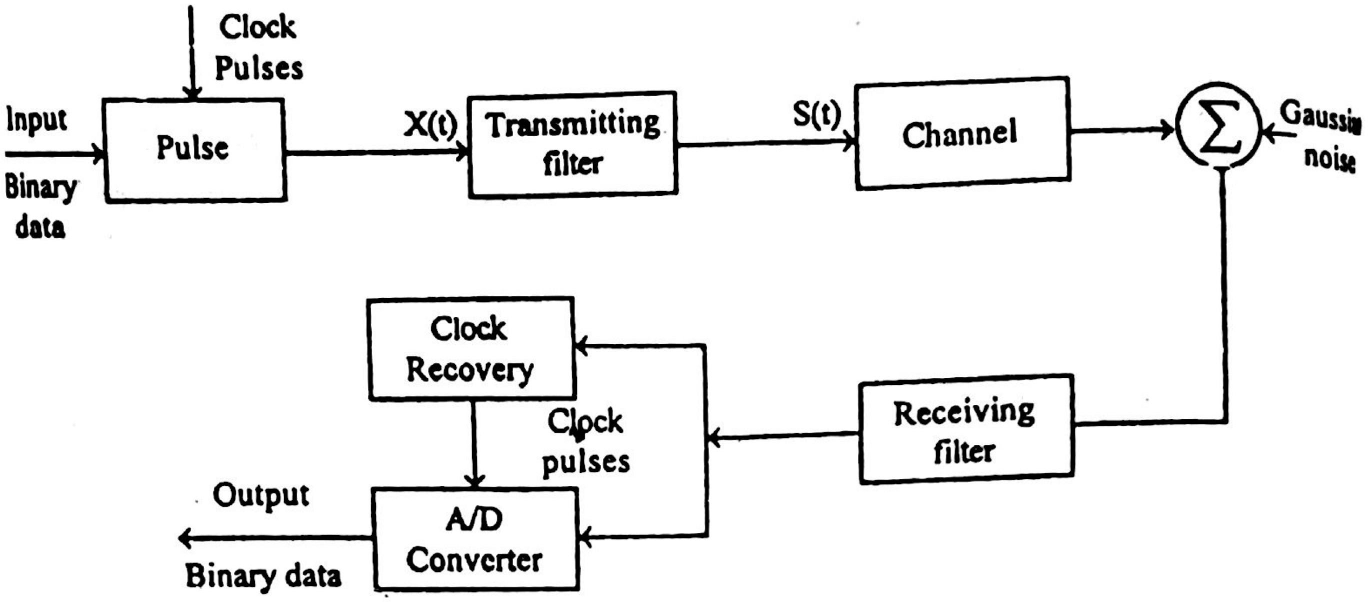


Figure 30 : Base band binary data transmission system

The sequence of short pulse produced by pulse generator is applied to a transmitting filter, its impulse response, 'g(t)' producing transmitted signal 's(t)'

$$s(t) = \sum_{K=-\infty}^{\infty} a_K g(t - KT_b)$$

The signal s(t) is modified as a result of transmission through the channel, having impulse response $h_1(t)$ result in which the channel adds random noise to the signal at the receiver input. The noisy signal is then passed through a receiver filter having its impulse response $h_2(t)$ and its output y(t) is sampled synchronously with the transmitter. The sampling time is determined by the clock or timing signal generated by the receiving filter itself. Then the sample sequenced is used to reconstruct the original data sequence by means by decision device.

The amplitude of each sample is compared to a threshold value 'λ'. If the amplitude of the sample exceeds the threshold value, the decision device 'Z' generates a symbol of '1'. If the amplitude of the sample is not exceeding threshold value, the decision device generates a symbol of '0'. If the amplitude of the sample equals the threshold value exactly then the receiver makes a guess to determine either '0' or '1' was transmitted.

The output of the receiver filter is given by

$$y(t) = \sum_{k=-\infty}^{\infty} \mu\, a_k\, p(t - kT_p) + n(t) \qquad (1)$$

where μ = scaling factor

$p(t)$ = combined impulse of the transmit filter, channel and receive filter

$$= g(t) \otimes h_1(t) \otimes h_2(t) \quad (2)$$

$p(t - kT_b)$ = delayed version of $p(t)$ by kT_b duration for k^{th} symbol in the sequence a_k.

$n(t)$ = additive white Gaussion noise

when $y(t)$ is sampled at time $t_i = iT_b$ then

$$y(t_i) = \mu \sum_{k=-\infty}^{\infty} a_k\ p(t_i - kT_b) + n(t_i) \quad (3)$$

since $t_i = iT_b$ then $y(t_i)$ becomes

$$y(t_i) = \mu \sum_{k=-\infty}^{\infty} a_k\ (iT_b - kT_b) + n(iT_b)$$

$$= \mu \sum_{k=-\infty}^{\infty} a_k\ (i - k)\ T_b + n(t_i) \quad (4)$$

The sample time t_i is synchronised with the transmitter clock. This means that the instant at which pulse a_k is transmitted is same as the time at which $y(t)$ is sampled. There is some delay during the transmission, however, for simplicity, the delay is assumed to be zero. i.e., the pulse is received as soon as it is transmitted.

hence the equation (4) becomes

$$y(t_i) = \mu\ a_i\ p(0) + \sum_{\substack{k=-\infty \\ k\neq i}}^{\infty} a_k\ p(i - k)T_b + n(t_i) \quad (5).$$

In equation (5) the first term represent the contribution of the ith transmitted bit. The second term represents the effect of all other transmitted bits on the decoding of i^{th} bit. This residual effect due to occurrence of pulses before and after the sampling instants 't_i' is called **"Inter symbol interference"** (ISI).

In the absence of ISI term in equation (5) then the output will be

$$y(t_i) = \mu a_i \quad (6)$$

If impulse response $p(t)$ is normalised, then $p(o) = 1$ and $\mu = 1$

thus $y(t_i) = a_i$

The equation (6) clearly shows that under ideal conditions, the ith transmitted bit is decoded correctly but in practice the ISI and noises are unavoidable in the system hence it introduces errors in the decision device at the receiver output.

The main objective of baseband PAM system design are to choose the transmitting and receiving filter to minimize the noise and ISI. In addition, for a given transmitted power it may be desirable to maximize the signalling rate 'r_b' for a given bandwidth or minimize the bandwidth required for a given signalling rate.

7. BINARY PHASE SHIFT KEYING (BPSK)

In a binary PSK system, binary symbol '1' and '0' modulate the phase of the carrier. Let us assume that the carrier is given as

$$S(t) = A \text{ Cos } 2\pi\phi_c t = \sqrt{2P_S} \text{ Cos } 2\pi f_c t$$

for symbol '1' is transmission

$$S_1(t) = \sqrt{2P} \cos 2\pi f_C t$$

If the next symbol '0' is transmission

$$S_2(t) = \sqrt{2P} \cos (2\pi f_C t + \pi) = -\sqrt{2P} \text{ Cos } 2\pi f_C t$$

since $\cos(\theta + \pi) = -\cos\theta$ thus in general

S(t) for BPSK can be written as

$$S(t) = b(t) \sqrt{2P} \cos 2\pi f_C t \qquad (1)$$

where $b(t) = +1$ for binary 1 is to be transmitted

$= -1$ for binary 0 is to be transmitted

Spectrum of BPSK signal

To obtain the frequency spectrum of BPSK signal

Take F.T on bothsides of equation (1) we get

$$S(f) = \frac{PT_b}{2}\left\{\left[\frac{S_{in}\ \pi(f-f_c)T_b}{\pi(f-f_c)T_b}\right]^2 + \frac{1}{2}\left[\frac{S_{in}\ \pi(f_c+f)T_b}{\pi(f+f_c)T_b}\right]^2\right\}$$

accordingly the spectrum is plotted as shown in figure 17.

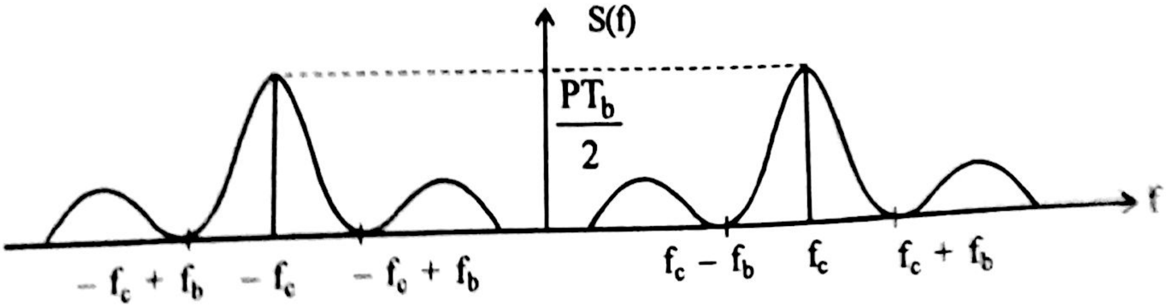


Figure 17

Bandwidth of BPSK signal

The spectrum of the BPSK signal in centred around the carrier frequency f_c as shown in figure 17(b). The main lobe is centred around the carrier frequency f_c and extends from $f_c - f_b$ to $f_c + f_b$, thus the bandwidth of BPSK signal is

$$BW = (f_c + f_b) - (f_c - f_b) = 2f_b$$

i.e., the minimum bandwidth of BPSK signal is equal to the twice of the highest frequency contained in base band signal.

- PSK is a form of digital angle modulation. It is similar to phase modulation in CW modulation. In this case there are two output phases, one phase represents logic 1 and other output phase represents logic, '0'. If the digital input signal changes its state, the phase of the output carrier shift between the two angles that are 180° out of phase.

- Block diagram of BPSK transmitter is shown in figure 17(b). The product or balanced modulator acts as phase reversing switch. Depending upon the input logic condition, i.e., '0' or '1', the carrier is transformed to the output either in phase or 180° out of phase with the reference carrier.

- In other words, in phase shift keying the phase of a carrier is switched between two values according to the two possible messages m_1 and m_2. The two phases are usually separated by π radians, hence it is also known as "**phase reversal keying**" (PRK). The binary phase shift keying is obtained through the system shown in figure 17(b).

- The effect of phase reversal modulation is to produce a double sideband suppressed carrier (square wave) amplitude modulated CW signal.

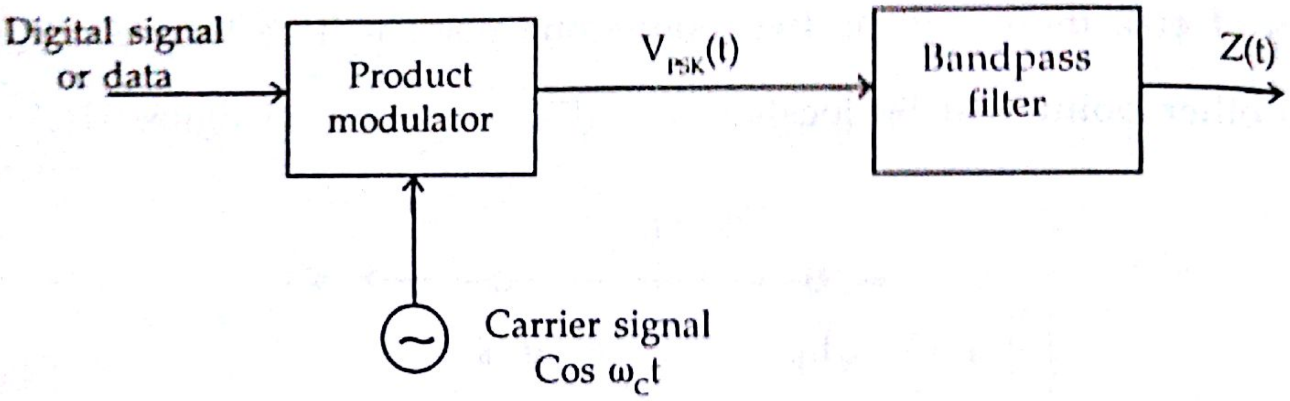


Figure 17(b) : PSK generation

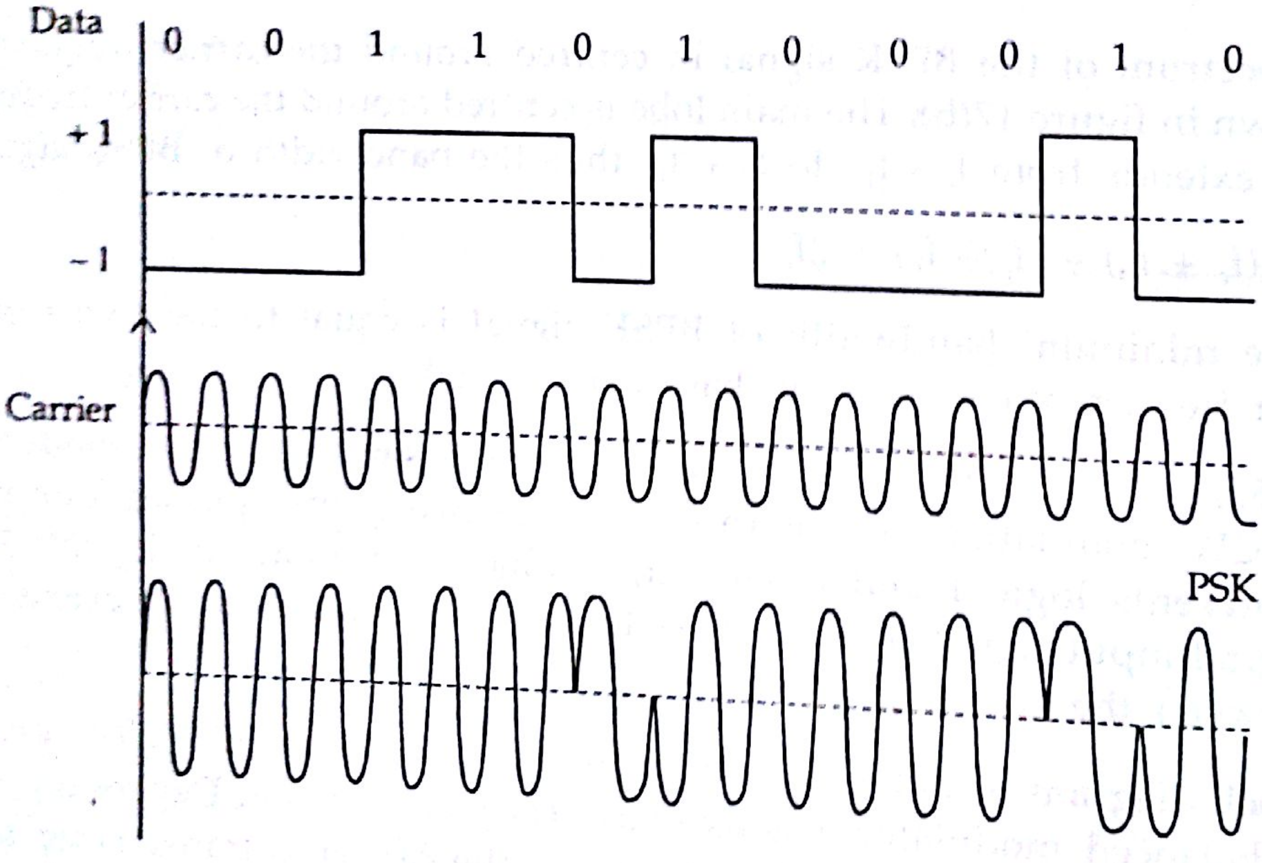


Figure 18 : Waveforms of Digital modulation schemes

In this case the signal or input data is translated into Non Return to Zero (NRZ) code having no dc component. The resulting pulses of amplitude ± Ac represent binary '1' and '0's. The output of the multiplier will be the PSK signal. It can be written as

$$V_{PSK}(t) = A \cos(\omega_C t + \phi) \text{ where } A = \sqrt{\frac{2E}{T_b}} = \sqrt{2P}$$

if ϕ = 0. then $V_{PSK}(t) = A \text{ Cos } \omega_C t = \sqrt{\frac{2E}{T_b}} \cos\omega_c t$ for $0 \le t \le T_b$

if $\phi = \pi$ then $V_{PSK}(t) = - A \text{ Cos } \omega_C t = -\sqrt{\frac{2E}{T_b}} \cos\omega_c t$ for $0 \le t \le T_b$

where E = Signal energy per bit

P = Power level

A = Peak amplitude of the carrier signal.

Coherent PSK Detection

The transmitted bit sequence can be recovered from the BPSK signal using correlation receiver shown in figure 19.

Let $S_1(t) = \sqrt{\frac{2E}{T_b}} \text{Cos}\omega_c t = A \text{ Cos}\omega_C t$ for $0 \le t \le T_b$ (for binary 0)

$S_2(t) = -\sqrt{\frac{2E}{T_b}} \text{Cos}\omega_c t = - A \text{ Cos}\omega_C t$ for $0 \le t \le T_b$ (for binary 1)

and $S_2(t) - S_1(t) = V_{PSK1}(t) - V_{PSK2}(t) = 2A \text{ Cos } \omega_C t = 2 \sqrt{\frac{2E}{T_b}} \text{Cos}\omega_c t$

It is synchronized in phase and frequency with the incoming signal. The signal components of the receiver output are,

$$S_{01}(t) = \int_0^{T_b} S_1(t)[S_2(t)-S_1(t)].dt$$

$$= \int_0^{T_b} A\text{Cos}\omega_c t\ [2A\text{Cos}\omega_c t]dt = \int_0^{T_b} 2A^2\text{Cos}^2\omega_c t\, dt$$

$$= \int_0^{T_b} 2A^? \left[\frac{1-\text{Cos}2\omega t}{2}\right] dt$$ neglect 2nd and higher order terms.

then $$= \int_2^{T_b} A^2\, dt = A^2 T_b = \frac{2E_b}{T_b}.T_b = 2E_b$$

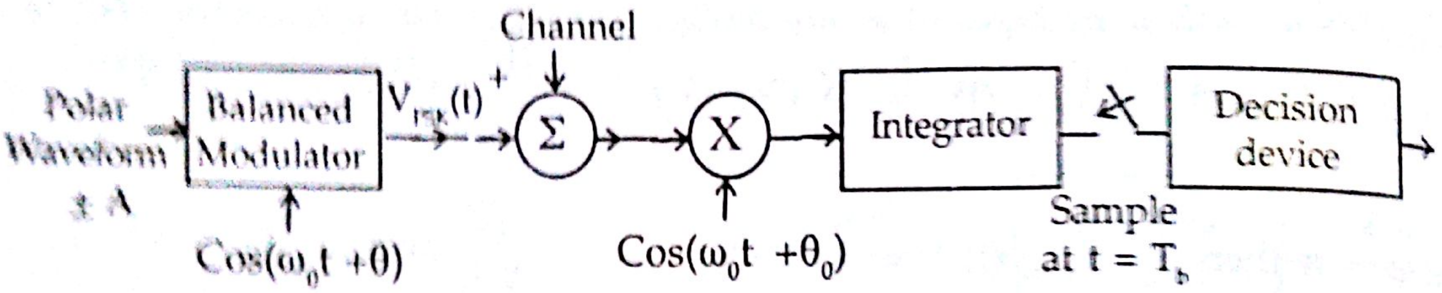


Figure 19 : Optimum coherent PSK system using correlators

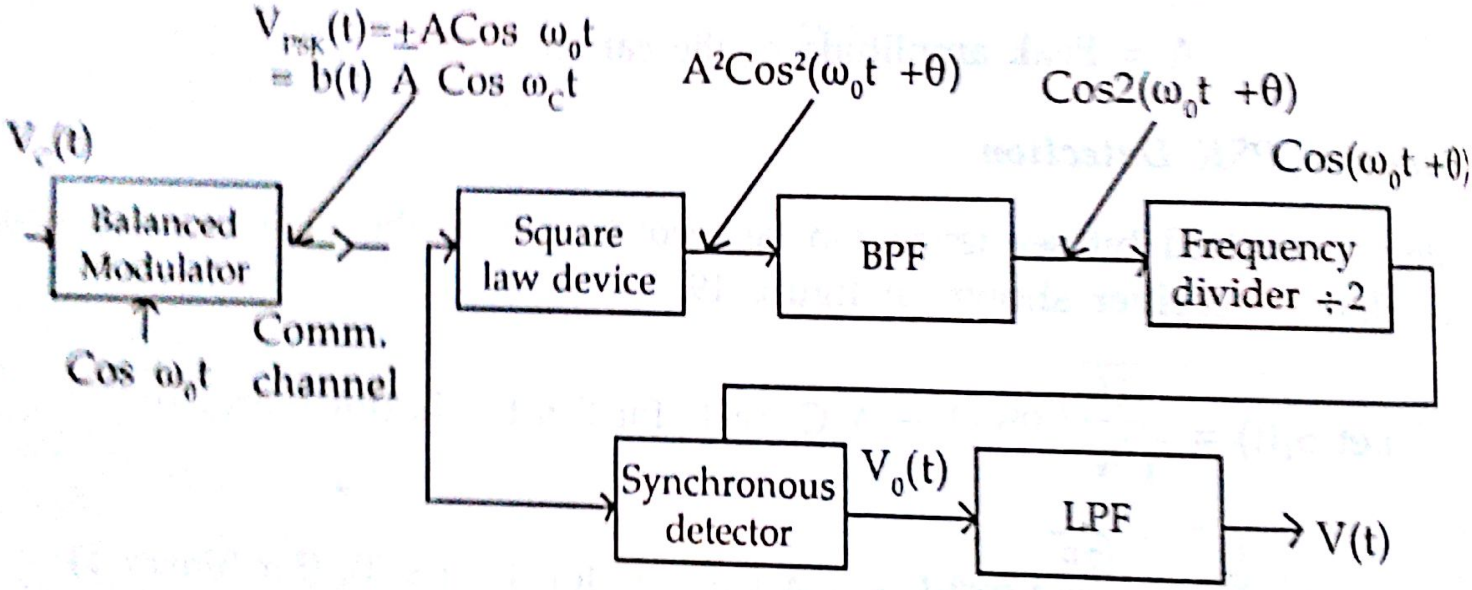


Figure 20 : A PSK system where b(t) = ± 1

A PSK system is shown in figure 20. The received signal is ± A Cos $(\omega_0 t + \theta)$. The detection is synchronous, hence a synchronous local carrier is necessary which is generated in the synchronising circuit.

8. QUADRATURE PSK

The prime requirement of the communication system is that transmission power and the channel bandwidth. In digital modulation the bandwidth is depends on the bit rate f_b. To reduce the bandwidth two or more bits are combined to one symbol results in bit rate is reduced. Thus, the frequency of the carrier needed also reduced, results in reduces the bandwidth of the channel. The QPSK is the one such a system, it is explained as follows. In QPSK two successive bits in data sequence are grouped together, this reduces the bit rate as well as channel bandwidth.

In BPSK, when the symbol changes the level, the phase of the carrier is changed by 180° because, there were only two symbols. However in QPSK, two successive bits are combined, there are four distinct symbols. When the symbol is changed to next symbol, then the phase of the carrier is charged by 45° refer Table 1.

In this case two binary PSK systems are used. The first PSK system having a carrier frequency of Cos $(\omega_0 t+ \theta)$ and second one is quadrature phase with first system. i.e. $\text{Sin}(\omega_0 t+ \theta_0)$. These two systems operate independently as long as ω_0 is an integral multiple of half the bit rate $\omega_0 = \left(\frac{m\omega_b}{2}\right)$ where m = 1, 2, . . . by combining the two into a single equivalent system result in which the bit rate is doubled over the channel (on the same carrier). Such a system is named as "**quadrature PSK or QPSK**" shown in figure 21.

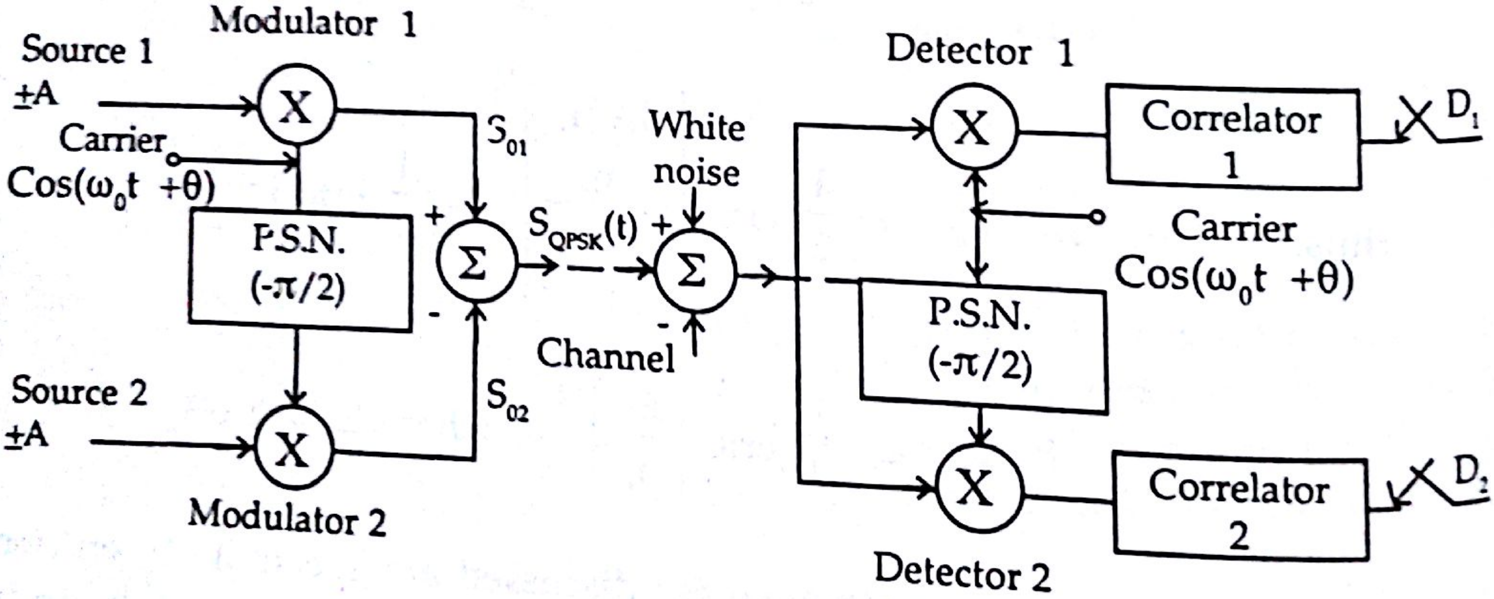


Figure 21 : QPSK System

The output of modulator 1 = ± A $\text{Cos}(\omega_0 t+ \theta_0)$

The output of modulator 2 = ± A $\text{Sin}(\omega_0 t+ \theta_0)$

Generation of QPSK

In this case the input binary sequence is first converted to a bipolar NRZ signal, it is denoted by S(t) refer figure 23. The demultiplexer divides S(t) into separate bit streams of the odd numbered $S_0(t)$ and even numbered bits $S_e(t)$ sequences. The symbol duration of both of these odd and even numbered sequences is $2T_b$, hence each symbol consists of two bits.

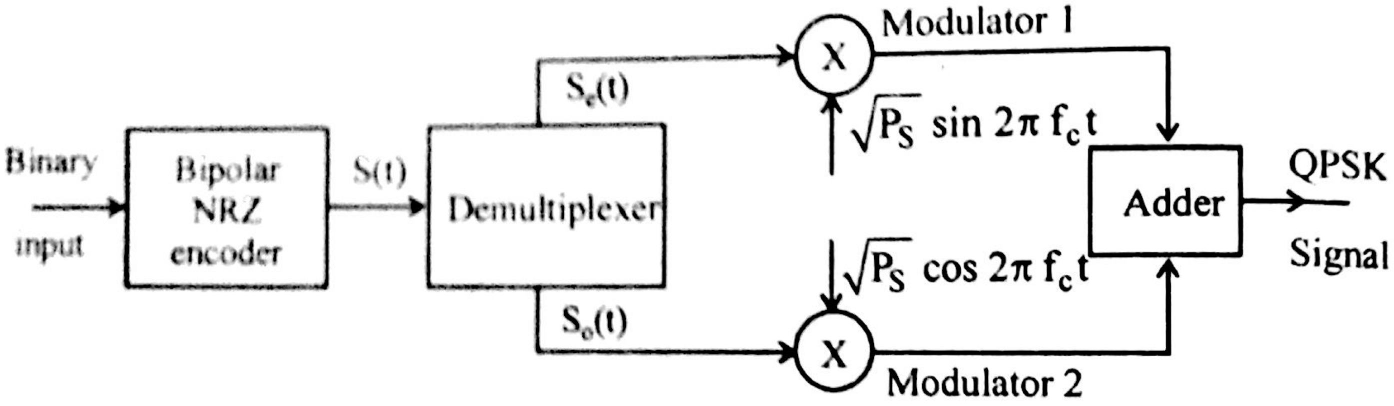


Figure 23: Generation of QPSK

It may be observed that, the first even bit occurs after the first odd bit, hence even numbered sequence $S_e(t)$ starts with the delay of one bit period T_b. This delay is known as **offset**. Hence is named as offset QPSK and it ensures the change in level of $S_e(t)$ and $S_o(t)$ cannot occur at the same time due to this offset.

Thus the output of modulator 1 = $S_e(t)\sqrt{P_S}$ Sin $2\pi f_C t = S_1(t)$

and the output of modulator 2 = $S_0(t)\ \sqrt{P_S}$ cos $2\pi f_c\ t = S_2(t)$

the value of $S_e(t)$ and $S_o(t)$ are +1 and −1 respectively.

The output of the adder is $S_{QPSK}(t) = S_1(t) + S_2(t)$

$$S_{QPSK}(t) = S_e(t)\sqrt{P_S}\ \sin 2\pi f_c t + S_o(t)\sqrt{P_S}\ \cos 2\pi f_c t \qquad (1)$$

Figure 23 shows the QPSK signal represented by equation (1). If there is any phase change, it occurs at minimum duration of T_b, due to the effect of offset.

Reception of QPSK

The figure shows the reception of coherent QPSK.

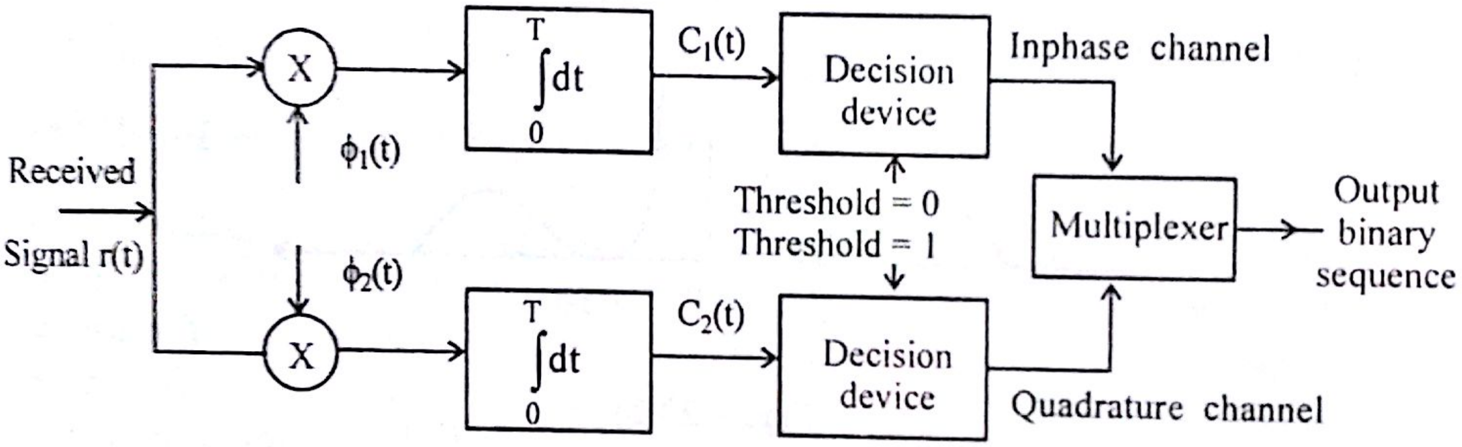


Figure 23(a)

The QPSK receiver consists of a pair of correlators with a common input and supplied with a locally generated carrier signals $\phi_1(t)$ and $\phi_2(t)$. The correlator outputs $C_1(t)$ and $C_2(t)$ are produced in response to the received signal r(t) are each compared with a threshold value. If $C_1(t) > 0$, a decision in made a symbol 1 for the inphase channel output, but if $C_1(t) < 0$, a decision is made symbol 0 for the inphase channel. Similarly if $x_2 > 0$, a decision is made, symbol 1 for the quadrature channel output and $x_2 < 0$, a decision is made, symbol 0 for quadrature channel output is obtained. Finally, these two outputs are combined in a multiplexer to reproduce the original binary sequence at the transmitter input with the minimum probability of error in an AWGN channel.

thus BW = $2\left(\frac{1}{2T_b}\right) = \frac{1}{T_b} = f_b$ thus the bandwidth of QPSK is reduced to half of the BPSK.

Advantages of QPSK

i. For the same bit error rate, the bandwidth required is reduced to half of the PSK.

ii. Due to the reduction of bandwidth, the information transmission rate is increased.

iii. Amplitude is remains constant thus carrier power is also remains constant.

9. DIFFERENTIAL PSK (DPSK)

- In a PSK system, although it is quite feasible to obtain a fixed phase recovered carrier, it is difficult to obtain the required absolute phase. The receiver must be given some indication of the proper phase reference since it is relatively easy for carrier recovery from the desired phase difference.

- Moreover the advantages of coherent PSK have been obtained at the cost of synchronous detection. Non coherent detection cannot be used because the information resides in phase. This difficulty is overcome in DPSK scheme.

- In this scheme, the information is encoded interms of phase changes between adjacent symbols, rather than an absolute phase for each symbol. Differential encoding of a message sequence is illustrated in table 3.

- An arbitrary reference binary digit is assumed for the initial digit of the encoded sequence. In the example shown in table 3 a '1' has been chosen. For each digit of the encoded sequence, the present digit is used as a reference for the following digit in the sequence.

- A '0' in the message sequence is encoded as a transition from the state of the reference digit to the opposite state in the encoded message sequence, '1' is encoded as no change of state. In the example shown, the first digit in the message sequence is a1, so no change in state is made in the encoded sequence and '1' appears as the next digit in the encoded sequence. This serves as the reference for the next digit to be encoded.

- Since the next digit appearing in the message sequence is '0', the next encoded digit is the opposite of the reference digit, or a '0'. The encoded message sequence then phase shift keys a carrier with the phase '0' and 'π' as shown in the table 3.

Table 3 Differential Encoding Example

Message sequence		1	0	0	1	1	1	0	0	0
Encoded sequence	1	1	0	1	1	1	1	0	1	0
Reference digit										
Transmitted phase	0	0	π	0	0	0	0	π	0	π

The block diagram shown in figure 24 illustrates the generation of DPSK. The equivalence gate, which is the negation of an Exclusive - OR, is a logic circuit that performs the operations listed in table. By a simple level shift at the output of the logic circuit, so that the encoded message is bi-polar, the DPSK signal is produced by multiplication by the carrier, or double side band modulation.

A possible implementation of a differentially coherent demodulator for DPSK is shown in figure 24. The received signal plus noise is correlated bit by bit with a one-bit delayed version of the signal plus noise. The output of the correlator is then compared with a threshold set at zero, a decision being made in favour of '1' or '0', depending on whether the correlator output is positive (or) negative respectively.

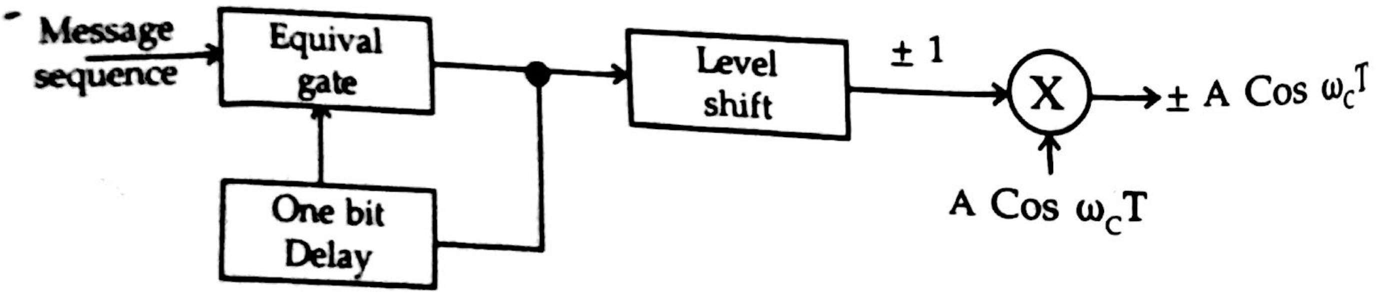


Figure 24 : Block diagram of a DPSK modulator

To illustrate that the received sequence will be correctly demodulated, consider the example given in the table 4. Assuming no noise is present. After the first two bits have been received (the reference bit plus the first encoded bit), the signal input to the correlator is $S_1 = A \text{ Cos } \omega_c t$ and the reference, or delayed, input is $R_1 = A \text{ Cos } \omega_c t$.

Table 4 : Truth table for the equivalence operation

Input 1 (Message)	Input 2 (Reference)	Output
0	0	1
0	1	0
1	0	0
1	1	1

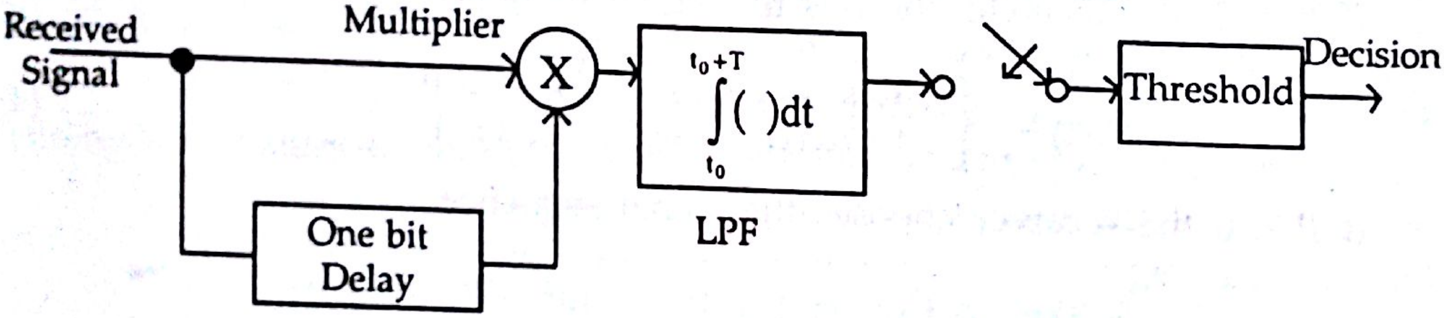


Figure 24(a) : Demodulation of DPSK

The output of the correlator is $V_1 = \int_0^{T_s} A^2 \text{Cos}^2 \omega_c t \, dt = \frac{1}{2} A^2 T$ . . . (1)

and the decision is that 1 was transmitted. For the next bit interval, the input are $S_2 = -A \text{ Cos } \omega_c t$ and $R_2 = S_1 = A \text{ Cos } \omega_c t$, resulting in a correlator output of

$$V_2 = \int_0^{T_s} (-A^2 \text{Cos}^2 \omega_c t) \, dt = -\frac{1}{2} A^2 T \quad \text{. . . (2)}$$

and a decision that '0' was transmitted. Continuing in this fashion, that the original message sequence is obtained if there is no noise at the input. This detector, is actually not optimum even it is simple to implement. The optimum detector for binary DPSK is shown in figure 25.

6

10. PERFORMANCE COMPARISON OF DIGITAL MODULATION SCHEMES

i. Bandwidth requirements: For high speed data transmission over a noisy bandpass channel VSB modulation with baseband signal shaping is better than ASK, FSK and PSK schemes for efficient bandwidth utilization. The BW of USB scheme is = γ_b. The BW of ASK and PSK = $2\gamma_b$ for FSK BW $\leq 2\gamma_b$. Thus if bandwidth is of primary factor, FSK is not used.

ii. If power requirement are most important, then coherent PSK or DPSK is most desirable while ASK are least desirable.

iii. If equipment complexity is a limiting factor, then non coherent demodulation schemes are preferable to coherent schemes.

Performance Measure of digital modulation schemes

Scheme	Input $S_1(t), S_2(t)$	B.W.	Probability of error	SNR dB	Equipment Complexity	Comments
Coherent ASK	$S_1(t) = A \text{ Cos } \omega_C t$	$2\gamma_b$	$\frac{1}{2} \text{erfc} \sqrt{\frac{\varepsilon}{2}}$	14.45	Moderate	Rarely used $V_T = \frac{A^2 T_b}{4}$
Non Coherent ASK	Same as above	$2\gamma_b$	$\frac{1}{2}\left[1+(2\pi\varepsilon)^{-1/2}\right] e^{-\varepsilon/2}$	18.33	Minor	$V_T = \frac{A}{2}$
Coherent FSK	$S_1(t) = A \text{ Cos } (\omega_C - \omega_d)t$ $S_2(t) = A \text{ Cos } (\omega_C + \omega_d)t$	$> 2\gamma_b$	$\frac{1}{2} \text{ erfc } \sqrt{\frac{\varepsilon}{2}}$	10.6	Major	Seldom used
Non Coherent FSK	Same as above	$> 2\gamma_b$	$\frac{1}{2} e^{-(\varepsilon/2)}$	15.33	Minor	Used for slow Speed date transmission
Coherent PSK	$S_1(t) = A \text{ Cos}\omega_C t$	$\cong 2\gamma_b$	$\frac{1}{2} \text{ erfc } \sqrt{\varepsilon}$	8.45	Major	Used for high speed data transmission
DPSK	Same as above	$\cong 2\gamma_b$	$\frac{1}{2} e^{-\varepsilon}$	9.30	Moderate	Most commonly

11. SYNCHRONIZATION TECHNIQUES

The synchronization is process used in a communication system to retrieve original message data effectively from the modulated signal.

When one or more signals are transmitted through the common channel using multiplexing technique, proper synchronization is required to detect the proper signal at the receiver. There are two basic modes of synchronization.

a. Carrier synchronization

If coherent detection is used in a receiver, then the knowledge of both the frequency and phase of the carrier is needed to recover the original message or data. The computation of frequency and phase of the carrier is called as **carrier synchronization.**

b. Symbol synchronization

If noncoherent detection is used to recover the message or data from the receiver it has to know the time at which the modulation can change its state. The computation of these times is called as **symbol synchronization.**

12. CARRIER SYNCHRONIZATION

One method of obtaining a carrier synchronization is to use the Costas detectors and shown in figure 26. It consists of two coherent or synchronous detectors. One detector is supplied with binary PSK and a locally generated carrier which is inphase with the transmitted carrier. This detector is known as "**In phase coherent detector or I channel**".

The other detector is feu with binary PSK and a locally generated carrier which is in quadrature phase with the transmitted carrier. This is known as "**Quadrature phase coherent detector or Q-Channel**". These two detectors are coupled together through VCO to form a negative feedback system designed in such a way to maintain the local oscillator to synchronise with the carrier.

Operation

- Assume the phase of local oscillator signal to be same as that of carrier i.e., local oscillator carrier signal is properly synchronised with transmitting carrier.
- In this case I channel output contains the desired demodulated signal whereas Q - channel output is zero due to the quadrature null effect of Q - channel.

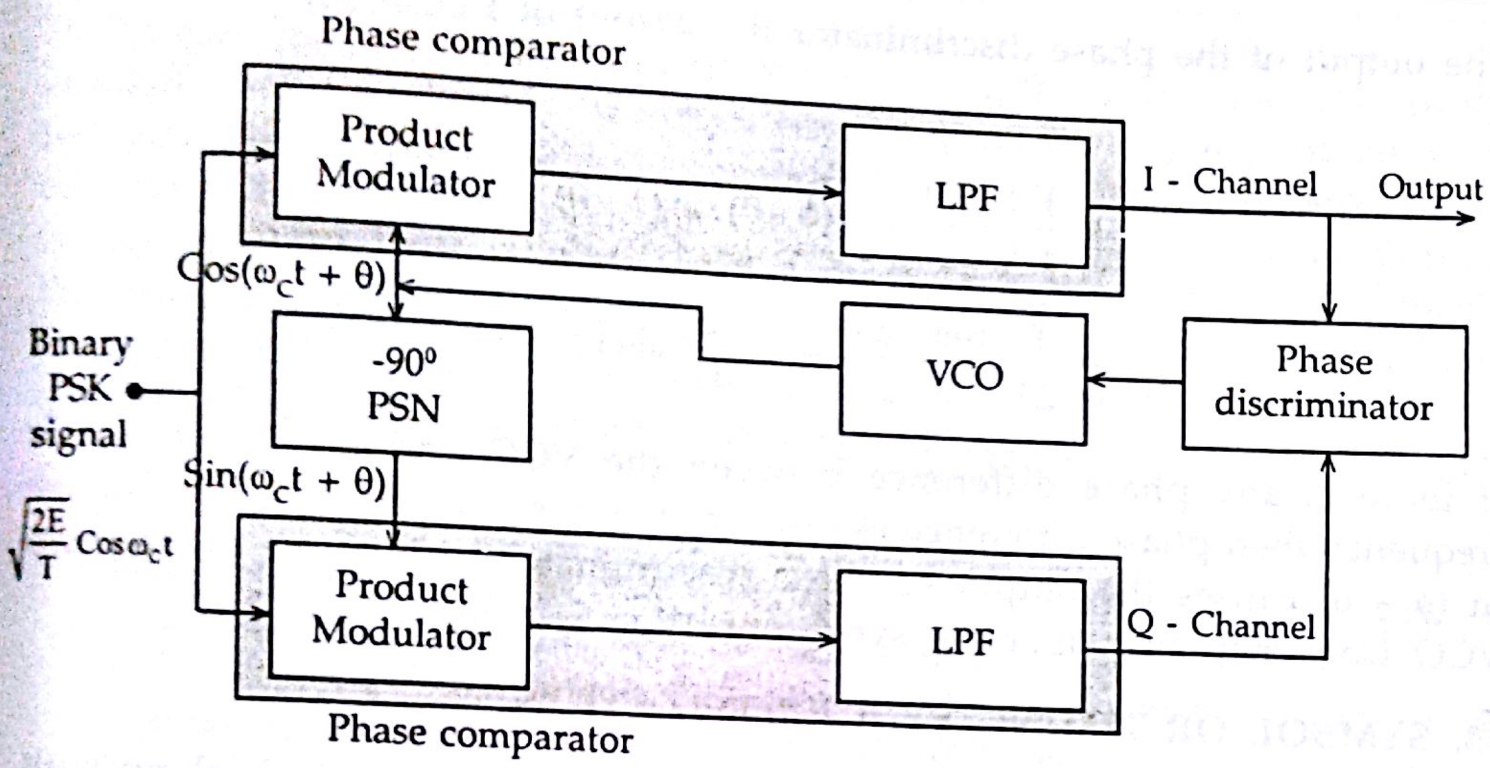


Figure 26 : Costas PLL Detector

- Suppose there is some phase shift 'ϕ' between local oscillator carrier and the transmitting carrier then 'I' channel output will remain in the same value but Q - channel output contains some signal which is proportional to sin ϕ.
- This Q - channel output will have same polarity as the I - channel output for one direction of local oscillator phase shift and opposite polarity for opposite direction of local oscillator phase shift.
- Thus combining the I and Q channel outputs in phase discrimination a dc signal is obtained that automatically corrects the phase errors in VCO. It is apparent that phase control in the costas detector ceases with modulation and that phase lock has to be re-established. This is not a serious problem when receiving voice transmission because the lock up process normally occurs so rapidly that no distortion is perceptible.

The input to costas loop is binary PSK = $\sqrt{\frac{2E}{T}} \cos \omega_c t$

The output of I channel = $\frac{1}{2}\sqrt{\frac{2E}{T}}\ Cos\,(\phi - \theta)$

The output of Q channel = $\frac{1}{2}\sqrt{\frac{2E}{T}}\ \sin\,(\phi - \theta)$

The output of the phase discriminator = (output of I channel) (Output of Q channel)

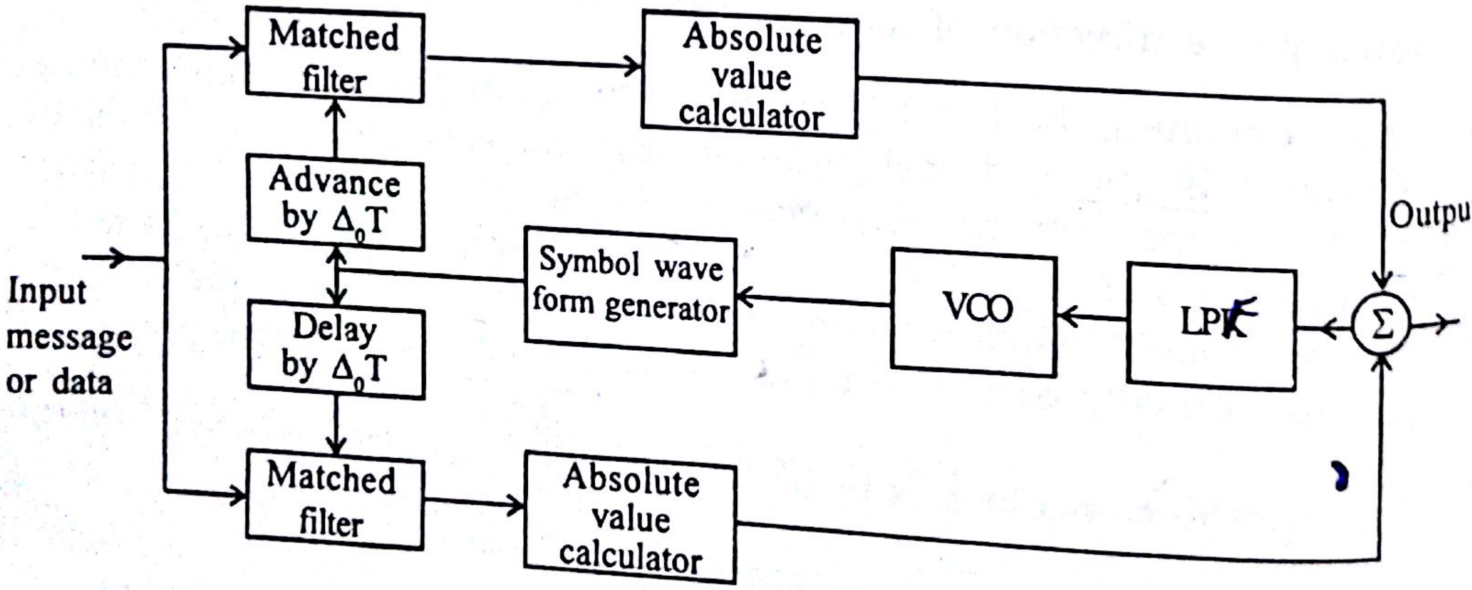
$$= \frac{1}{4}\left(\frac{2E}{T}\right) \text{Cos}\,(\phi-\theta)\,\sin\,(\phi-\theta)$$

$$= \frac{E}{2T}\,\frac{\text{Sin}\,2(\phi-\theta)}{2} = \frac{E}{4T}\,\sin\,(\phi-\theta)$$

If there is any phase difference between the VCO and the input carrier frequency then phase difference (ϕ - θ) is changed proportionally. The change in (ϕ – θ) causes the output of phase discriminator to increase or decrease VCO frequency such that the synchronization is achieved.

8 13. SYMBOL OR TIMING SYNCHRONIZATION

The symbol synchronization is obtained by transmitting a clock along with the message or data signal in multiplied form. At the receiver, the clock is extracted by appropriate filtering of the modulated waveform. Such as approach minimizes the time required for clock recovery. The main draw back of this method is that, a fraction of the transmitted power is needs to allocated to the transmission of the clock. To avoid this problem matched filters are used at the receiver. The figure 27 shows the **early late gate type symbol synchronization**. We know that the matched filters in maximum at the sampling time.



In this case the error signal is generated by taking the difference between the absolute values of the two matched filter outputs for a given off set (α T) between the actual transmission times t(KT) and this local estimates $\hat{t}$ (KT), otherwise, it is linearly proportional to 'α' respective of the polarity. If there is any error, it is low pass filtered and then applied to a VCO that controls the charging and discharging instants of matched filters. The instantaneous frequency of the local clock is advanced or retarded in an iterative manner until the equalization is reached, thus symbol synchronization is obtained.

2. DEFINITIONS OF SPREAD SPECTRUM

- Spread spectrum is a modulation technique whereby a modulated waveform is modulated second time in such a way so as to generate an expanded bandwidth (wideband) signal, that does not significantly interfere with other signals. Bandwidth expansion is achieved by a second modulation and it is independent of message or information transmitted.

- It also can be defined as, spread spectrum is a technique in which a Pseudo noise code, independent of the information data, is employed as a modulation waveform to spread the signal energy over the bandwidth much greater than the signal information bandwidth. At the receiver the signal is despread using a synchronised replica of the Pseudo noise code.

- Spread spectrum means spreading the bandwidth of the signal to be transmitted than the minimum bandwidth necessary to transmit it. It is achieved by second time modulating the modulated signal using Pseudo noise code as carrier signal. The same Pseudo noise carrier is used at the receiver to despread (demodulate) the signal. As a result of this, it is able to reject the interference caused by other user either intentionally or unintentionally.

- The spread spectrum signals used for the transmission of digital information are distinguished by the characteristic that their bandwidth 'W' is greater than the information rate R bits/sec. i.e., the bandwidth expansion factor $\left(B_e = \frac{W}{R}\right)$ for a spread spectrum signal is much greater than unity. The large redundancy inherent in spread spectrum signals is required to overcome the interference encounted over some radio and satellite channels. It is achieved by using suitable coding techniques, thus that coding is the important element in the design of spread spectrum modulation.

- A second important element employed in the design of spread spectrum signal is pseudo-randomness, it makes the signals appear similar to random noise and difficult to demodulate the receivers other than intended ones.

Uses of spread spectrum communication

i. Suppressing the determined effects due to jamming, interference arising from other users of the channel, and self interference due to multipath propagation.

ii. Achieving message privacy in the presence of other listeners.

iii. Hiding a signal by transmitting it at low power, and thus making it difficult for a unintended receiver to detect in the presence of noise.

Advantages of spread spectrum modulation

i. Improved interference rejection

ii. Code division multiplexing for CMDA application.

iii. Low density power spectra for signal hiding.

iv. High resolution ranging

v. Secure communications.

vi. Antijam capability

vii. Increased capacity and spectral efficiency in mobile communication systems.

viii. Lower cost of implementation

ix Readily available IC components.

Applications of spread spectrum

a). Military communication systems

It has two functions

i) It allows a transmitter to transmit message to a receiver without the message being detected by unauthorised receivers.

ii) To achieve this the spread spectrum modulation decreases the transmitted power spectral density to an unauthorised receiver.

b). In commercial communication system, the spread spectrum signals are transmitted over an already existing microwave signal, with the same carrier frequency, resulting in which additional signals can be transmitted over the same band thereby increasing the number of users.

c). It is used in satellite communication and local area networks.

d). Multiple access communications in which a number of independent users are required to share a common channel without external synchronization mechanism. Example : ground based mobile radio environment involving mobile vehicle that must communicate with a central station.

The spread spectrum techniques are classified as follows

5. PRINCIPLE OF DIRECT SEQUENCE SPREAD SPECTRUM (DSSS)

Input

Binary data d_t with symbol rate $R_S = 1/T_S$ (=bit rate R_b for BPSK)

Pseudo - noise code PN_t with chip rate $R_C = 1/T_C$ (an integer of R_S)

The data sequence represented by d_K, is converted to bipolar NRZ wave form d(t) such as follows

if $d_K = 1$ then $b(t) = 1$

and $d_K = 0$ then $b(t) = -1$

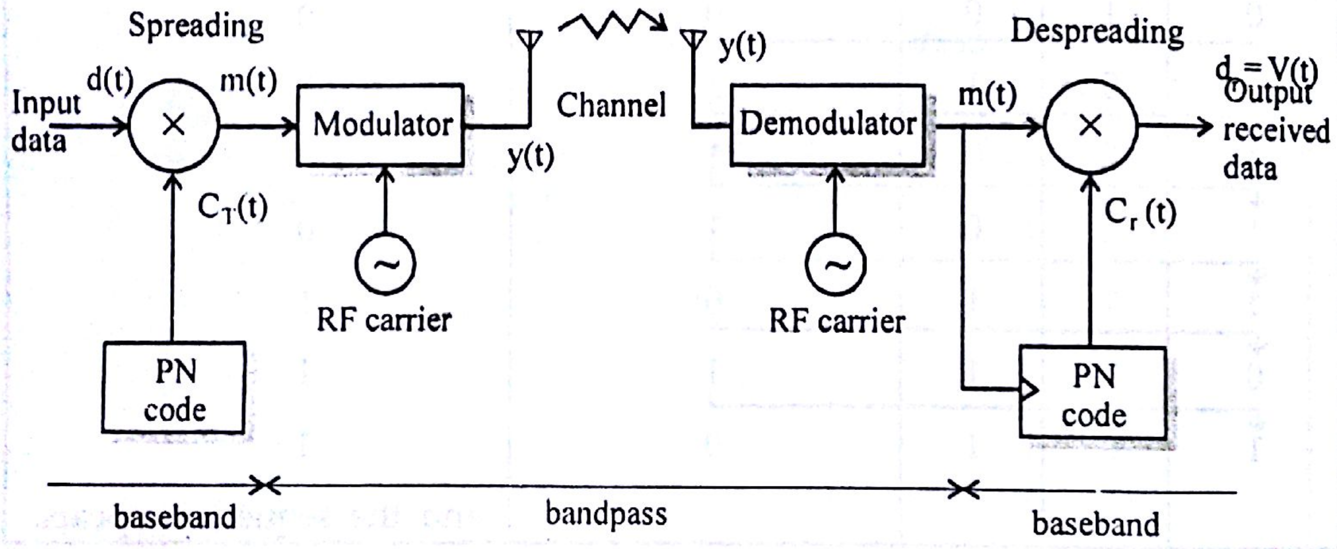


Figure 6

Similarly the PN sequence C(t) is represented as $C_K = 1$ then $C(t) = 1$

and $C_K = 0$ then $C(t) = -1$.

Modulation (Spreading)

In the transmitter the binary data d_t or d(t) (for BPSK, I and Q for QPSK) is directly multiplied with the PN sequence C(t) which is independent of the binary data, to produce the transmitted baseband signal M(t).

$$M(t) = d(t)\ C(t)$$

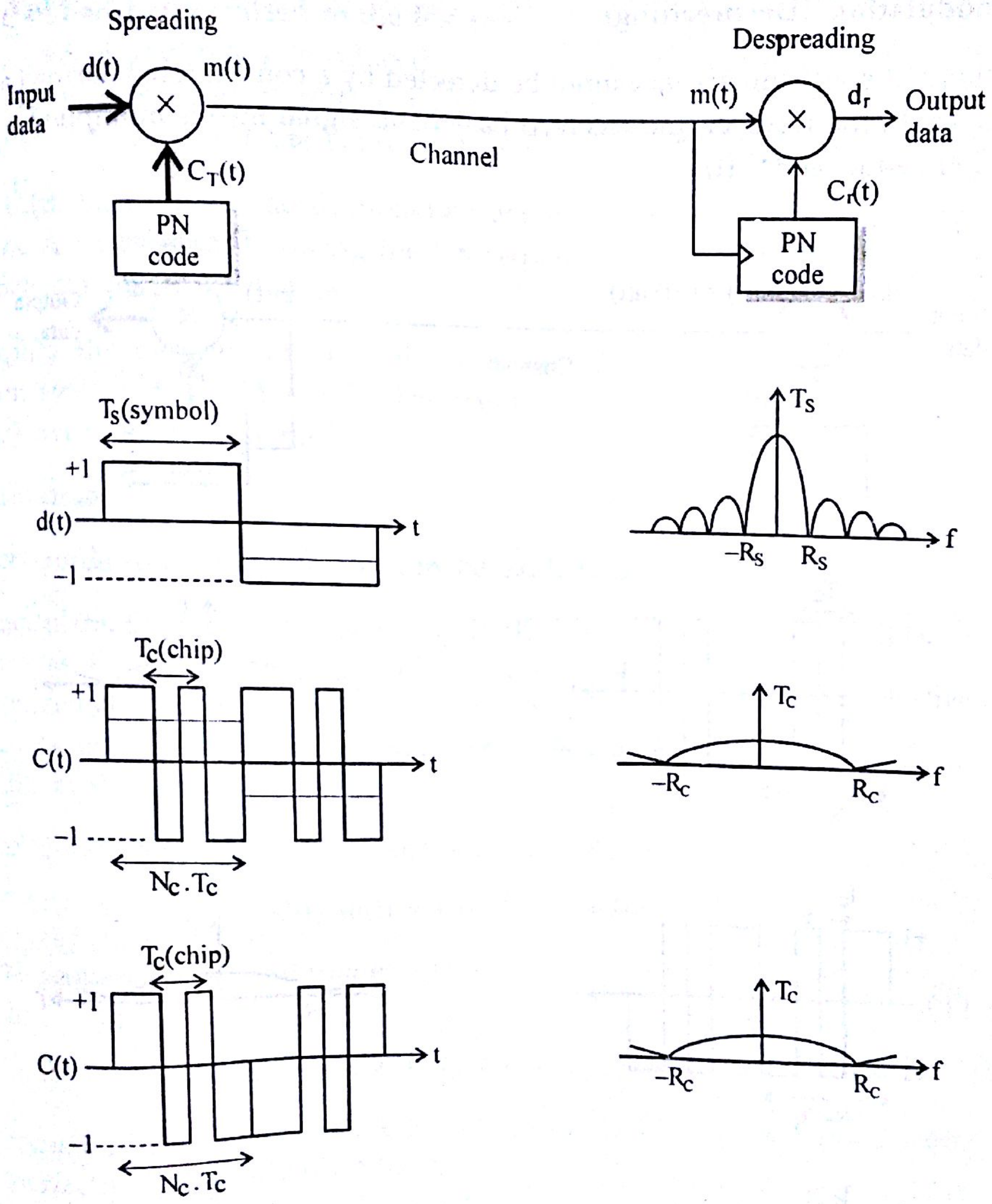


Figure 7 : DSSS (Spreading)

The effect of multiplication of d_t with a PN sequence is to spread the base band bandwidth R_S of d_t to a baseband bandwidth of R_C refer figure 7.

Demodulation (Despreading)

The spread spectrum signal cannot be detected by a conventional narrowband receiver. In the receiver, the received base band signal m(t) is multiplied with the PN sequence $C_r(t)$

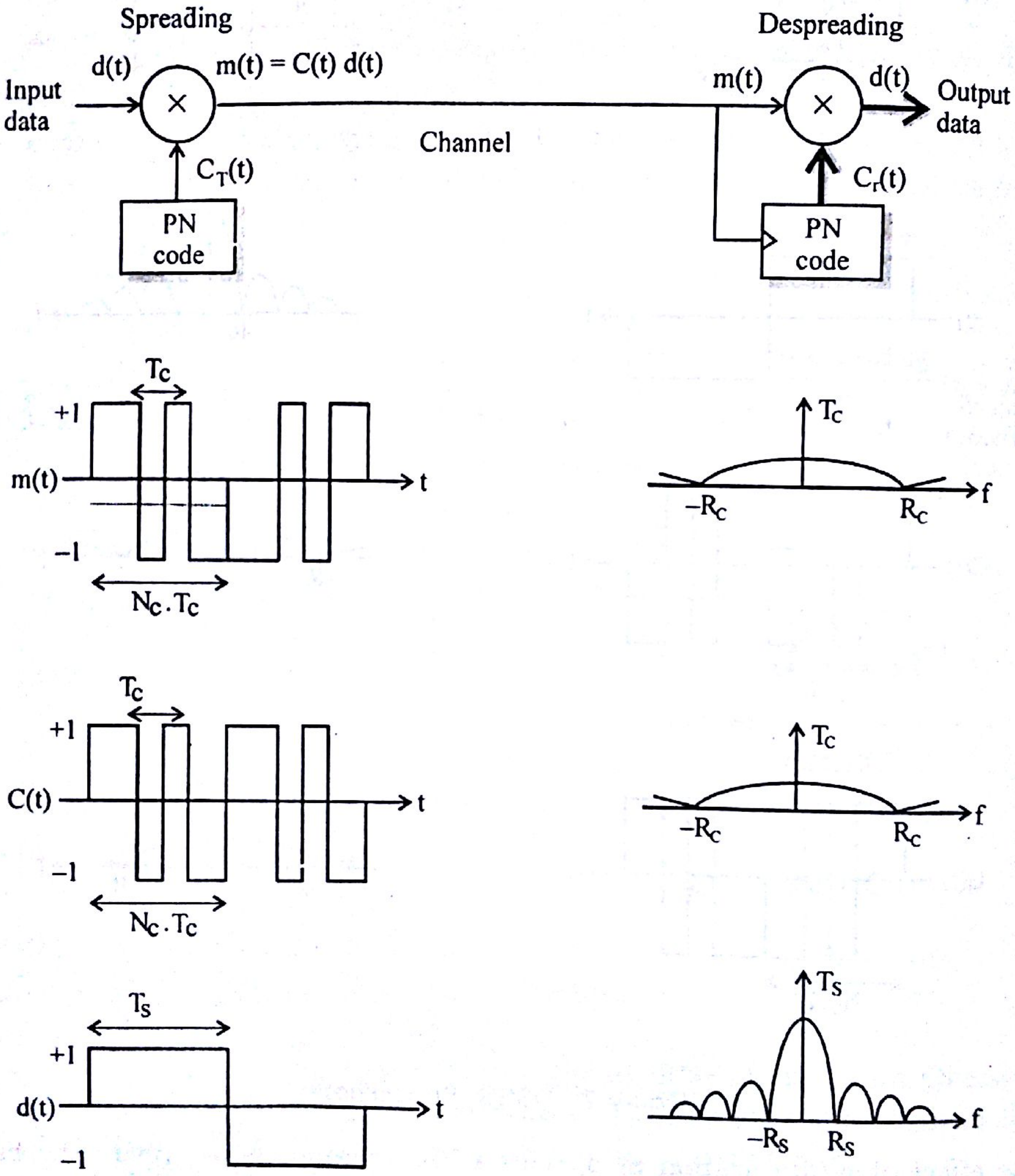


Figure 8 : DSSS (Despreading)

$C_T(t) = C_r(t)$ and synchronized to the PN sequence in the received data, than the recovered binary data is produced on d_r. The effect of multiplication of the spread spectrum signal M(t) with the PN sequence $C_T(t)$ used in the transmitter is to despread the bandwidth of M(t) to R_S refer figure 8.

$C_T(t) \neq C_r(t)$, than there is no despreading action. The signal d_r has a spread spectrum. A receiver not knowing the PN sequence of the transmitter cannot reproduce the transmitted data.

To simplify the description of modulation and demodulation, the spread spectrum system is considered for baseband BPSK communication (without filtering) over an ideal channel.

Demodulation

In other words, the demodulation can be explained as follows

To demodulate, the received signal is multiplied by $C_r(t)$ this is the same PN sequence as $PN_t = C_T(t)$ (the Pseudo-noise code used in the transmitter), synchronized to the PN sequence in the received signal $C(t)_b$. This operation is called (spectrum) despreading, since the effect is to undo the spreading operation at the transmitter.

The multiplier output in the receiver is then $C_T(t) = C_r(t) = C(t)$

$$d_r = C(t)\ m(t) = C(t)\ [C(t)\ .\ d(t)]$$

The PN sequence $C_T(t)$ alternates between the levels – 1 and +1, in the example:

$$C_T(t) = +1 + 1 + 1 - 1 + 1 - 1 - 1$$

The alternation is destroyed when the PN sequence $C_T(t)$ is multiplied with itself (perfectly synchronised), because

$$C_T(t)\ .\ C_r(t) = +1 \quad \text{for all } t$$

Thus autocorrelation $R_a\ (\tau = 0)$ = average $(C_T(t)\ .\ C_r(t) = +1$

The data signal is reproduced at the multiplier output

$$\text{(data received)}\quad d_r = d_t\quad \text{(data transmitted)}$$

If the PN sequence at the receiver is not synchronized properly to the received signal, the data cannot be recoverd.

If C(t) of transmitter ≠ C(t) of receiver

If the received signal is multiplied by a PN sequence $C_r(t)$, different from the one used in the modulator, the multiplier output becomes

$$d(t) = m(t)\ C_r(t) \quad [d(t)\ .\ C_T(t)]\ C_r(t)$$

In the receiver, detection of the desired signal is achieved by correlation against a local reference PN sequence. For secure communications in a multi-user environment, the transmitted data d_t may not be recovered by a user that doesn't know the PN sequence $C_T(t)$ used at the transmitter. Therefore

Crosscorrelation $R_C(\tau) = \text{average}\ C_T(t)\ C_r(t) << 1$ for all τ

is required. This orthogonal property of the allocated spreading codes, means that the correlator used in the receiver is approximately zero for all except the desired transmission.

9. FREQUENCY HOPPING SPREAD SPECTRUM

- In DS-SS system the use of PN sequence is to modulate a PSK signal and it achieves instantaneous spreading of the transmission bandwidth. The ability of such a system to suppress the effects of jamming or interfering is determined by the processing gain of the system, which is a function of the PN sequence length.

- The processing gain can be made larger by employing a PN sequence with narrow chip duration, which inturn permits a greater bandwidth and more chips per bit. But, the capabilities of physical devices used to generate the PN sequence imposes practical limitation on the processing gain. Hence attaining the larger processing gain is practically impossible. To over come this problem frequency hop spread spectrum technique is used.

- In a frequency hopped spread spectrum system the available channel bandwidth is divided into a large number of continuous frequency slots. In any signalling the transmitted signal occupies one or more of the available frequency slots, the selection of the frequency slots in each signalling interval is made Pseudo randomly according to the output from PN generator.

- The frequency hopping does not cover the entire spread spectrum instantaneously, hence we are led to consider the rate at which the hops occur. There are two types of frequency hopping.

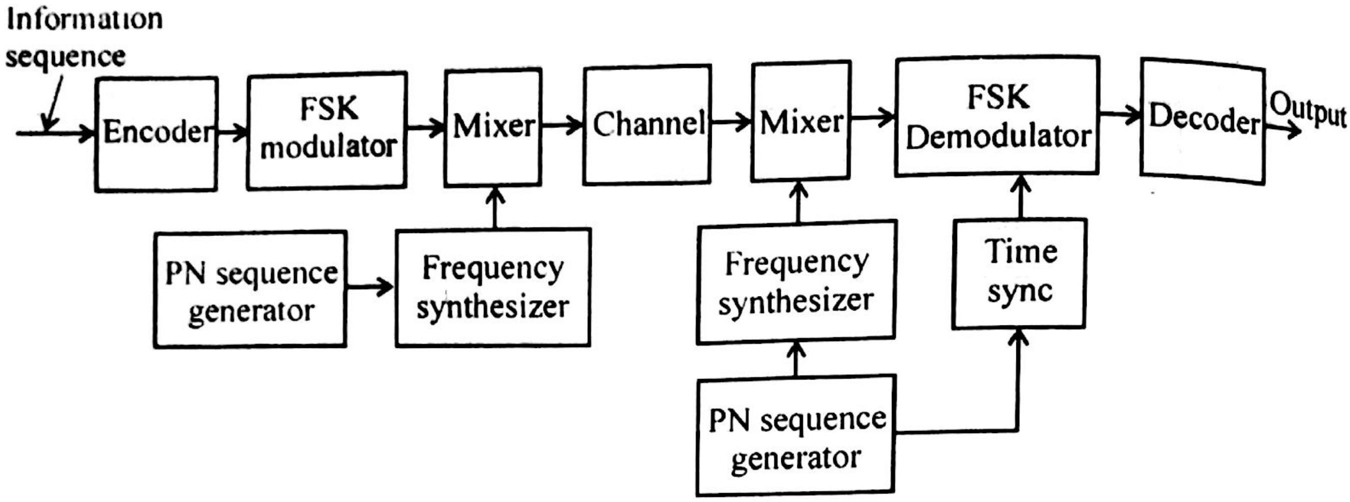


Figure 14 : Block diagram of FH spread spectrum system

A block diagram of the transmitter and receiver for a frequency hopped spread spectrum system is shown in figure 14. The modulation is usually either by binary or 'M' ary FSK. Suppose, the binary FSK is employed the modulator selects one of the two frequencies corresponding to the transmission of '0' or '1'. The resulting FSK signal is translated into frequency by an amount that is determined by PN generator, which inturn is used to select the frequency that is synthesized by the frequency synthesizer.

The output of frequency synthesizer and FSK modulator is mixed in this unit and the resultant frequency translated signal is transmitted over the channel. For (example) 'M' bits obtained from PN generator may be used to specify 2^M-1 possible frequency translation.

10. SLOW FREQUENCY HOPPING SS

In which the symbol rate R_S of 'M' ary FSK signal is an integral of the hop rate R_h. i.e., several symbols are transmitted on each frequency hop. Each symbol of slow R_S, R_h and R_C for slow frequency hopping that

$$R_S = R_C = \frac{R_b}{K} \geq R_h$$

where R_b = bit rate of incoming binary data

$$K = \log_2 M.$$

At the receiver, the PN sequence are identical and synchronised with the received signal, which is used to control the output of the frequency synthesizer.

Thus the Pseudo random frequency translation introduced at the transmitter is removed at the receiver by mixing the synthesizer output with the received signal. The resultant signal is demodulated by means of a FSK demodulator. A signal for maintaining synchronism of the PN generator with the frequency translated received signal is usually extracted from the received signal.

The PSK modulation gives better performance than FSK modulation in additive white gaussian noise channel, but, it is difficult to maintain phase coherence in the synthesis of frequencies used in hopping and the transmitted frequency hopped signal. Consequently FSK modulation with non coherent detection is usually employed with FH spread spectrum signals.

On a single hop, the bandwidth of the transmitted signal is the same as that of conventional 'M' ary FSK signals. However, for a complete range of 2^K frequency hops the FH-M FSK signal occupies a much larger bandwidth. The bandwidth of FH is in the order of several GHz which is very much larger than DS SS.

Let K successive bits of input data sequence represent $2^K = M$ symbols frequency hop is nothing but frequency slot. The rate of change of frequency slot or hop is known as hop rate R_h. The rate at which 'K' bit symbols of data input sequences are generated are called as symbol rate R_S.

The frequency hopping rate R_h may be equal to symbol rate or lower or higher than the symbol rate. If R_h is equal to or lower than the symbol rate then the FH system is named as "**slow hopping FHSS**".

If R_h is higher than symbol rate then the FH systems are named as "**fast hopping FHSS**".

An individual FH/FSK signal of short duration is refened to as a chip. The chip rate R_C for an FH/FSK SS is $R_C = \max(R_h, R_S)$.

At each hop FSK signals are separated in frequency by an integer multiple of chip rate $R_C = R_S$. This condition ensures that any transmitted symbol will not produce any crosstalk in it. But the performance of slow FHSS system is same as that of non coherent detection of conventional MFSK signal in AWGN. Thus the interfering signal has an effect on the FH/MFSK receiver interms of average probability of symbol error.

If the jammer spread its average power 'J' over the entire frequency hopped spectrum, the jammer's effect is equivalent to an AWGN with power spectral density $\frac{N_0}{2}$ where $N_0 = \frac{J}{W_C}$ and W_C bandwidth of FHSS signal.

Energy to noise density ratio $\frac{E}{N_0} = \frac{P/J}{W_C/R_S}$

we know P/J is the reciprocal of jamming margin.

and processing gain $PG = \frac{W_C}{R_S} = 2^K$

$PG \text{ in db} = 10\log_{10} 2^K = 10K \log_{10}2 = 3K \text{ db}.$

K = length of PN sequence used.

11. FAST FREQUENCY HOPPING SS

In which the hop rate R_h is an integer multiple of the M ary FSK signal rate R_S. i.e., the carrier frequency will change or hop several times during transmission of one symbol.

A fast FH/MFSK system differs from a slow FH/MFSK system is as follows. In a fast FH/MFSK system, each hop is a chip. In general fast frequency hoping is used to detect a smart jamming or interference signal. It has two operations (i) measuring the spectral content of the transmitted signal, and (ii) retuning of the interfering signal to that position of the frequency band. To overcome the jamming, the transmitted signal must be hopped to a new carrier frequency before the jamming is able to complete the processing of these two functions.

For data recovery at the receiver, non coherent detection is used, but the detection procedure is quite different from that used in slow FH/MFSK receiver. In particular,

1. For each FH/MFSK symbol, separate decision are made on the 'K' frequency hop chips received and a simple rule based on majority is used to make and estimate of the dehopped MFSK signal.
2. For each FH/MFSK symbol likelihood functions are computed as function of the total signal received over 'K' chips and the larger one is selected.

The codes used in practical applications are almost linear codes.

7

11. BLOCK CODES

Block codes also known as "**arithmetic codes**", in which each block of 'K' message is encoded into a block of 'n' bits. (n > k) as shown in figure 12. The check bits are derived from the message bits and are added to them. The n bit block of channel encoder output is called a "**codeword**" and the codes in which the message bits appear at the beginning of a codeword are called "**systematic codes**".

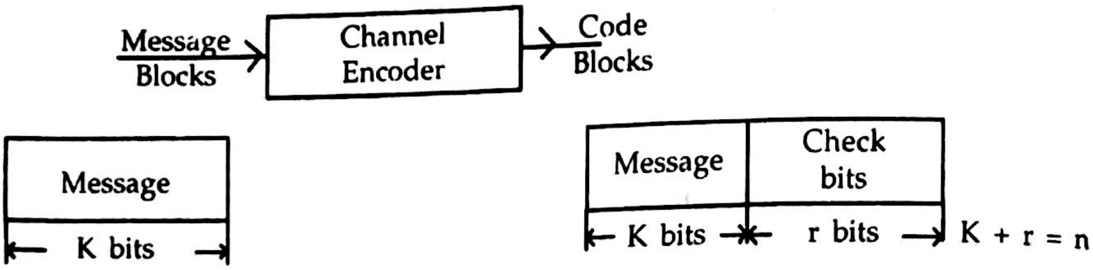


Figure 12

Parity check code: The simplest possible block code is when the number of check bits is one. These are known as parity check codes. When the checkbit is such that the total number of 1's in the codeword is even, it is known as **"Even Parity Check"** and when the check bit is such that the total number of 1's in the codeword is odd, it is known as **"odd parity check"**.

Message	**Codeword for parity message**	**even check**	**Code for message**	**odd parity checkbit**
010011	010011	1	010011	0
101110	101110	0	101110	1

If a single error occurs in a received message it can be immediately detected, although the erroneous bit cannot be determined. Thus with this code, though a single error can be detected, it cannot be corrected.

Study of Binary code space

The "**weight of a codeword**" is defined as the numbers of non zero components in it. For example:

Codeword	**Weight**
010110	3
101000	2

The "**Hamming distance**", between two codewords is defined as the number of components in which they differ

Mathematically, the hamming distance can be defined as

$$D(u,v) = \sum_{K=1}^{n} (\alpha_K \oplus \beta_K)$$

where $u = \alpha_1, \alpha_2 \ldots\ldots\ldots\ldots\ldots\ldots \alpha_n$

$v = \beta_1, \beta_2 \ldots\ldots\ldots\ldots\ldots\ldots \beta_n$ (where α and β's are '0' or 1)

The notation ⊕ means modulo - 2 addition for which the rules are

$$0 \oplus 0 = 0$$
$$0 \oplus 1 = 1$$
$$1 \oplus 0 = 1$$
$$1 \oplus 1 = 0$$

then ie. u = 1010 and v = 0111 given.

$D(u, v)\ [(1 \oplus 0) + (0 \oplus 1) + (1 \oplus 1) + (0 \oplus 1)] = [1+ 1 + 0 + 1] = 3.$

The minimum distance of a block code is defined as the smallest distance between any pair of codewords in the code.

Let us consider a block code of two digits with a minimum distance two. Two code books are possible. The codes are 00, 01, 10, 11; code book values are 01, 10.

Now, a data is received with single error, 01 may be received either as 00 or as 11. These values are not available in our codebook, hence an error has been detected. But a decision cannot be taken as to whether 01 or 10 was transmitted, because both are at equal distance from 00 hence error cannot be corrected.

If the codebook of minimum distance three the single error can be detected as the distance of erroneous word is 1 from only one codeword, and more than 1 from all other code words.

For example : if 000, 111 is our code book values. If 001 is received, a decision can be taken that 000 is received since the distance between 000 and 001 is one. Whereas the distance between 111 and 001 is two.

Therefore the minimum hamming distance should be two to detect error, and it should above two for correct errors.

Minimum distance	Description of coding
1	Error cannot be detected
2	Single error detections
3	Single error detection and correction
4	Single error correction plus double error correction
5	Double error correction
6	Double error correction plus triple error detection

In general, if 'n' is the minimum distance of a block code then

(i) $\frac{n-1}{2}$ errors can be corrected if 'n' is odd.

(ii) $\frac{n-2}{2}$ errors can be corrected and n/2 error can be detected if 'n' is even.

$x_4 = d_1 \oplus d_3$; $x_5 = d_1 \oplus a_2$; $x_6 = d_2 \oplus d_3$

cf

14. CYCLIC CODES

Cyclic code is a subclass of linear block codes. An advantage of cyclic code over other types of code is that they are easy to encode. The cyclic codes possess well defined mathematical structure, which has led to the development of very efficient decoding schemes for them. There are two important reasons to use cyclic codes.

i) Encoding and syndrome calculations can be easily implemented by using simple shift registers with feedback connections.

ii) the mathematical structure of these codes is such that it is possible to design codes having useful error correcting properties.

A binary code is said to be a "**cyclic code**" if it exhibits two fundamental properties.

a) Linearity : The sum of any two codewords in the code is also a codeword.

b) cyclic property : Any cycle shift of a code word in the code is also a code word.

Property (a) restates that the cyclic code is a linear blockcode.

Property (b) in mathematic terms, let the 'n' tuple $(C_0, C_1, \dots C_{n-1})$ is a code vector of 'C' The code is cyclic if the 'n' tuples.

$$
\begin{array}{c}
(C_0, C_1 \dots C_{n-1}) \\
(C_{n-1}\ C_0 \dots C_{n-2}) \\
(C_{n-2}\ C_{n-1} \dots C_{n-3}) \\
\vdots \quad \vdots \\
(C_1\ C_2 \dots C_{n-1}\ C_0)
\end{array}
$$

i.e., shifting each code cyclically one place to the right.

Example : 1 0 1 1 the codes are 1011, 1101, 1110, 0111

1 1 0 1 obtained by a cyclic shift of

1 1 1 0 'n' tuple 1011 (n = 4)

0 1 1 1

1 0 1 1

The code obtained by rearranging the four words is also a cyclic code.

The codeword 'C' can be represented by a code nomial as

$$C(X) = C_0 + C_1X + C_2\,X^2 + \dots + C_{n-1}\,X^{n-1} \qquad \dots (1)$$

Where X = arbitrary real variable.

The coefficients of the polynomials are '0' and 1's and they belong to a binary field which satisfies the following rules of additions and multiplication.

$$0 + 0 = 0 \qquad 0 . 0 = 0$$
$$0 + 1 = 1 \qquad 0 . 1 = 0$$
$$1 + 0 = 1 \qquad 1 . 0 = 0$$
$$1 + 1 = 0 \qquad 1 . 1 = 1$$

Each power of 'x' in the polynomial (X) represents a one bit cyclic shift in time. Hence multiplication of the polynomial C(X) by X is viewed as cyclic shift, subject to the constant $x^n = 1$. For a single cyclic shift, thus equation (1) becomes

$$C(X) = C_{n-1} + C_0 X + C_1 X^2 + \ldots + C_{n-2} D^{n-1} \text{ and so on.}$$

Generator polynomial

If g(x) is a polynomial of the degree (n – k) and is a factor of $x^n + 1$, then g(x) generates (n, k) cyclic code in which the code polynomial C(x) for a data vector $D = (d_0, d_1, d_2 \ldots d_{K-1})$ is generated by

$$C(X) = D(X) . g(X).$$

Consider 'K' polynomials g(X) xg(x), $x^2g(x) \ldots x^{K-1} g(x)$ which all have degree n – 1 or less. For any linear combination of these polynomials of the form $C(X) = d_0 g(X) + d_1 X g(X) + \ldots + d_{K-1} X^{K-1} g(X) = D(X) g(X)$.

where C(X) is a polynomial of degree n – 1 or less and is a multiple of g(x). There are 2^K polynomials corresponding to 2^K message data vectors.

The codeword corresponding to the 2^K polynomials form a (n, k) linear code and is also cyclic.

Proof:

Let $C(X) = C_0 + C_1X + C_2X^2 + \ldots + C_{n-1} X^{n-1}$ . . . 1

for a single cyclic shift

$$C^1(X) = C_0X + C_1X^2 + C_2X^3 + \ldots + C_{n-2} X^{n-1} + C_{n-1} \quad \ldots 2$$

$$XC(X) = C_{n-1}X^n + C_0X + C_1X^2 \ldots + C_{n-2} X^{n-1} \quad \ldots 3$$

The Equation (2) and (3) are added by mod '2' adder, thus

$$C^1X + XC(X) = C_{n-1}X^n + (C_{n-2} + C_{n-2})\,X^{n-1} + \ldots + (X_1 + X_1)\,P^2 + (X_0 + P_0)X + C_{n-1}$$

$$C^1X + X(X) = C_{n-1}(X^n+1)$$

[Since both bits are same then the result is zero in mod 2 addition].

Since there is no difference in mod '2' addition and subtraction we get.

$$= C_{n-1}\ (X^n + 1) + (C_{n-1} + C_0X + \ldots. + C_{n-2}X^{n-1})$$

$$= C_{n-1}\ (X^n+1)+C^{(1)}(X)$$

where $C^{(1)}(X)$ is a cyclic shift of C(X). Since XC(X) and X^n+1 are both divisible by g(x). Thus $C^{(1)}(X)$ is a multiple of g(X) and can be expressed as a linear combination of g(X), Xg(X), $X^2g(X)$ . . . X^{K-1} g(X). i.e., $C^{(1)}(X)$ is also a code polynomial.

The generator polynomial g(x) of a cyclic code can be written into a systematic form such as follows.

$$C(x) = \underbrace{P_0\ P_1\ P_2 \ldots P_{n-K-1}}_{n-K \text{ Parity check bit}}\ \underbrace{d_0\ d_1\ \ldots d_{K-1}}_{\text{'K' message bit}} \qquad \ldots (1)$$

where $P(X) = P_0+P_1X+P_1X^2+\ldots\ldots+P_{n-K-1}X^{n-K-1}$

= parity check polynomial for the message signal D(X)

= remainder from dividing X^{n-K} D(X) by g(x)

i.e., $X^{n-K}\,D(X) = q(x)g(x) + P(x)$ . . . (2)

where P(X) and q(X) are the remainder and quotient respectively thus the code polynomial is written as

$$C(x) = P(x) + X^{n-K}D(x) \qquad \ldots (3)$$

Encoder for Cyclic code

The block diagram representation of cyclic codes are shown in figure. The encoding is described by the equation 1, 2 and 3 involve the division of X^{n-K} D(X) by the generator polynomial g(x) to calculate the parity check polynomial P(x). It can be obtained by using the dividing circuit consisting of a feedback shift register as shown in figure 15.

The hardware required to implement this encoder are,

(i) An (n – K) bit shift register.

(ii) Maximum of (n – K) mod 2 adder.

(iii) AND gate as a switch.

Symbols used are → $\boxed{P_0}$ → = flipflop

$\oplus$ = module 2 adder;

→ ⓖ → g = 1 provide closed path

= 0 provide open path

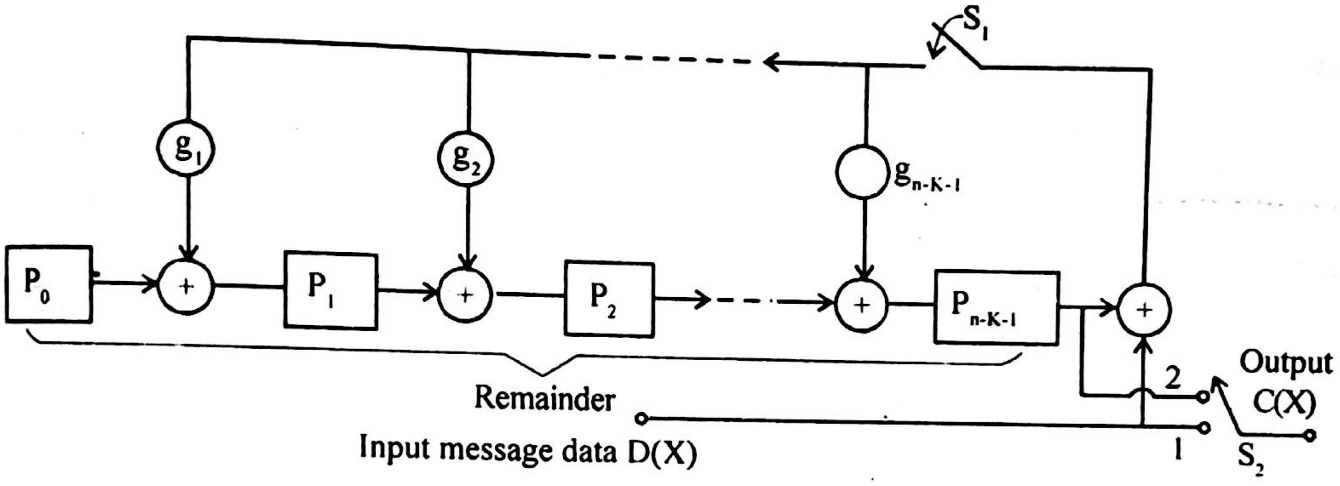


Figure 15 : Encoder for cyclic code

Operation

i. When the switch 'S' is switched ON and the switch S_2 is in position 1 all the shift registers are initialized to zero state. Now the message data (d_0 d_1 . . .d_{K-1}) are shifted into the register and simultaneously into the communication channel.

As soon as the 'K' information digits have been shifted into the register, the register contains (n – K) parity check bits (P_0 P_1 . . . P_{n-K-1}).

ii. When the switch S_1 is tuned OFF, and the switch S_2 is in position '2' now the contents of the shift register are shifted into the channel. Thus the code word (P_0 P_1 P_2. . . P_{n-K-1} d_0 d_1 d_2. . . d_{K-1}) is generated and sent over the channel.

This encoder is much simpler than the encoder needed for implementing (n, K) linear block codes, where positions of the 'G' and 'H' matrix have to be stored.

Syndrome calculation

Let C(x)= codeword transmitted over the noisy channel.

R= Received vector, it may not be any one of 2^K valid code

If the syndrome is zero, the received vector is divisible by the generator polynomial thus it is valid code then the decoder accepts the received vector as the transmitted code vector.

If syndrome is not zero then the transmission error have occurred. The syndrome 'S' of the received code R(X) is the remainder resulting from dividing R(X) by g(X).

i.e.,
$$\frac{R(X)}{g(X)} = Q(X) + \frac{S(X)}{g(X)} \qquad \ldots 1$$

where Q(X) = Quotient of the division

$$R(X) = C(X) + E(X) \qquad \ldots 2$$

E(X) = Error caused by the channel.

thus
$$\frac{R(X)}{g(X)} = \frac{C(X)+E(X)}{g(X)} = \frac{C(X)}{g(X)} + \frac{E(X)}{g(X)} = C(X) + \frac{E(X)}{g(X)} \qquad \ldots 3$$

we know C(X) = D(X).g(X) hence from equation (1) and (3) we get

hence
$$C(X) \oplus \frac{E(X)}{g(X)} = Q(X) + \frac{S(X)}{g(X)}$$

or
$$\frac{E(X)}{g(X)} = C(X) + Q(X) + \frac{S(X)}{g(X)} \quad \text{or} \quad E(X) = [C(x) + Q(X)]\, g(X) + S(X)$$

[note : All addition are mod '2' addition operation]

Hence the syndrome of R(X) is equal to the remainder resulting from dividing the error pattern by the generator polynomial and the syndrome contains information about the error pattern that can be used for error correction.

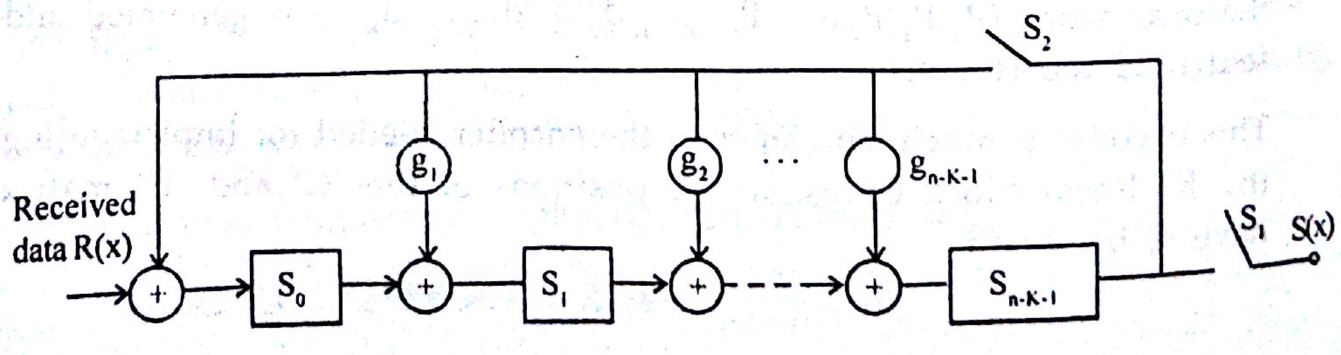


Figure 16 : Syndrome decoding of cyclic code

Steps to be followed

i. First of all, the registers are initialised. Then the switch S_2 switched ON and S_1 switched OFF, now the received vector R(X) entered into the shift register, after the entire received code is entered into the register, the content of the register will be syndrome.

ii. Now switch S_2 is opened and S_1 is closed, thus the contents of shift registers are shifted out and the register is ready for processing the next received vector.

Once the syndrome is calculated, then an error pattern is detected for that particular syndrome. When this error vector is added to the received vector 'Y' then it gives corrected code vector at the output.

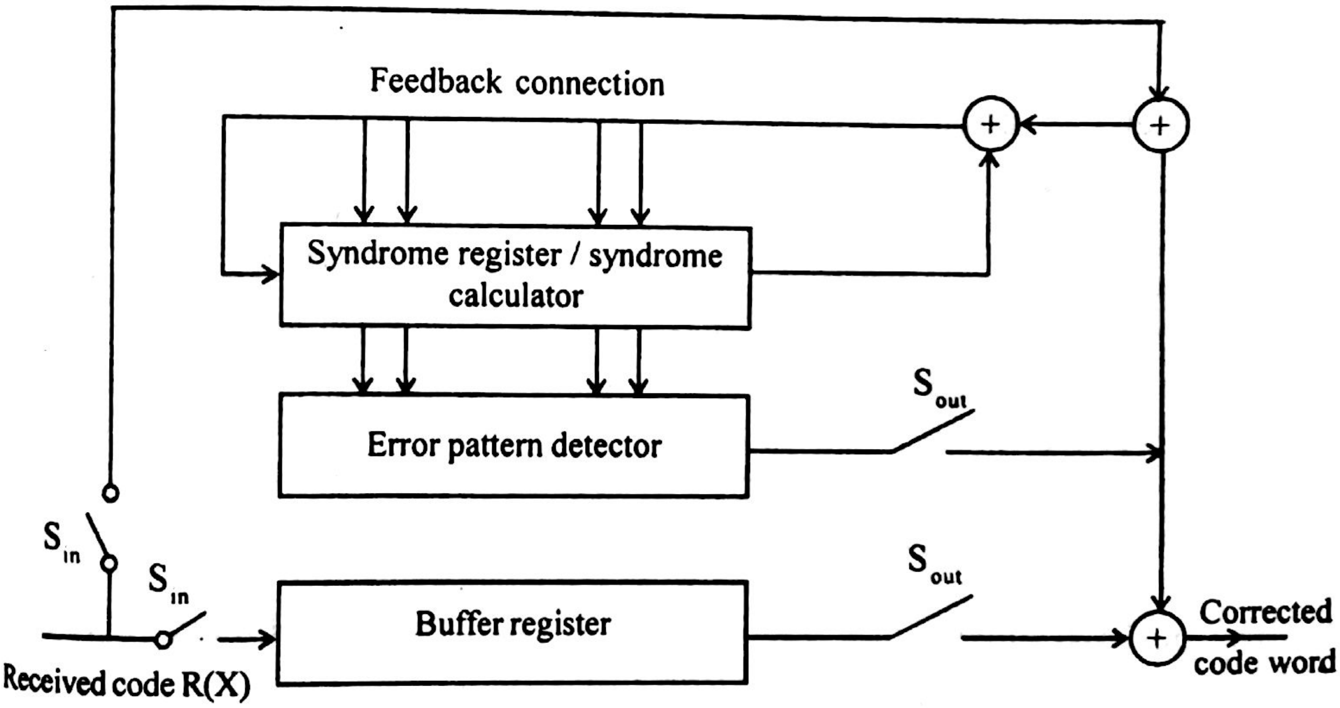


Figure 17

Operation

i. Initially switches S_{in} are closed and S_{out} are opened, thus the bits of the received vector 'R' shifted into buffer register as well as they are shifted into the syndrome calculator.

ii. After the syndrome for the received vector is calculated and is placed in the syndrome register itself. Now the contents of the syndrome register is read by error detector. The detector output is 1 if and only if the syndrome in the syndrome register corresponds to a correctable error pattern. With an error the highest order position (X^{n-1}) and so on. The decoder operates on the received code digit by digit until the entire received code is shifted out of the buffer.

iii. The switch S_{in} are opened and S_{OUT} are closed, now the shifts are applied to the flipflops of buffer register, error register and syndrome register. The error pattern is then added digit by digit to the received code, if the syndrome register contains all zeros then the output is error free code word, otherwise the output is erroneous.

Advantages

i. Error correction and detection is very simple and easy to implement.

ii. It avoids the storage of matrix 'G' and 'H' or look up table decoding.

iii. Encoder and decoder are simple to implement.

Disadvantages

i. Error detector circuit is complex to implement.