DIGITAL SIGNAL PROCESSING

SUBJECT CODE-BEE604

YEAR-III

SEMESTER –VI

INTRODUCTION TO DIGITAL SIGNAL PROCESSING

WHAT IS DSP ?

DSP is the processing of signals by digital means.

In general terms, a signal is a stream of information representing anything from stock prices to data from a remote-sensing satellite.

In many cases, the signal is initially in the form of an analog electrical voltage or current, Produced for example by a microphone or some other type of transducer.

In some situations the data is already in digital form. eg: o/p from the readout System of a CD Player.

An analog signal must be converted into digital form before DSP techniques can be applied.

An analog electrical voltage signal can be digitized using an analog to Digital converter.

An analog signal on Sampling results in a discrete signal followed by quantization and encoding in order to convert the discrete signal to digital signal.

Signals need to be processed in a variety of ways. For example, the output signal from a transducer may well be contaminated with noise.

Processing the signal using a filter circuit can remove or atleast reduce the unwanted part of the signal.

DSP technology is commonly employed nowadays in devices such as mobile phones, multimedia computers, video recoders, CD players, hard disk drive controllers and modems, and will soon replace analog circuitry TV sets and telephones.

An important application of DSP is in signal compression and decompression.

In CD systems, for example, the music recorded on the CD is in compressed form (to increase storage capacity) and must be decompressed for the recorded signal to be reproduced.

Signal compression is used in digital cellular phones to allows a greater no. of calls to be handled simultaneously within each local "cell".

Although the mathematical theory underlying DSP techniques such as FFT, Wavelet transform, Hilbert transform, Digital filter design and signal compression can be fairly complex, the numerical operations required to implement these techniques are in fact very simple.

The architecture of DSP chip is designed to carry out such operations incredibly fast, processing upto tens of millions of samples per second, to provide real time performance.

In signal processing, the function of a filter is to remove unwanted parts of the signal. There are two main kinds of filters, analog and digital.

An analog filter uses analog electronic circuits made from components such as resistors, capacitors and op-amps to produce the required filtering effect.

Such filter circuits are widely used in applications such as noise reduction, video signal enhancement, graph equalizers in hi-fi systems and many other areas.

A digital filter uses a digital processor to perform numerical calculations on sampled values of the signal. The processor may be a general purpose computer such as PC, or a specialized DSP chip.

The main advantages of digital filters over analog filters are listed below.

1. A digital filter is programmable
2. Digital filters are easily designed, tested and implemented on a general purpose computer or workstation.
3. The characteristics of analog filter circuits are subject to drift and are dependent on temperature. Digital filters do not suffer from these problems, and so are extremely stable with respect to both time and temperature.
4. Unlike their analog computers counterparts, digital filters can handle low frequency signals accurately. As the speed of DSP technology continues to increase, digital filters are being applied to high frequency signals in the RF domain, which in the past was the exclusive preserve of analog technology.
5. Digital filters are very much more versatile in their ability to process signals in a variety of ways. i.e) Digital filter adapt to changes in the characteristics of

UNIT - I

SIGNALS AND SYSTEMS

SIGNALS

A Signal is defined as a function of one or more variables and conveys information.

A signal is a physical quantity that varies with time in general, or any other independent variable.

i) ONE - DIMENSIONAL SIGNAL

When a function depends on a single variable to represent the signal, it is said to be 1D signal.

Eg: ECG signal, Speech signal.

ii) TWO - DIMENSIONAL SIGNAL

When a function depends on two variables to represent the signal, it is said to be 2D signal.

Eg: Video Signal

iii) MULTIDIMENSIONAL SIGNAL

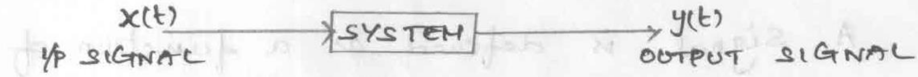
When a function depends on more than two variables to represent the signal, it is said to be a multidimensional signal.

Eg: 3-Dimentional representation of images.

SYSTEMS

A system is defined as a physical device that that performs an operation on a signal.

A system is an entity that manipulates one or more input signals to perform a function, which results in a new output signal.



BASIC SYSTEM

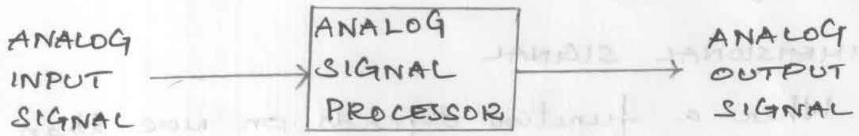
Eg:
A filter used to reduce the noise and interference currupting a desired information bearing signal

BASIC ELEMENTS OF DSP

Passing the signal through the system is known as processing.

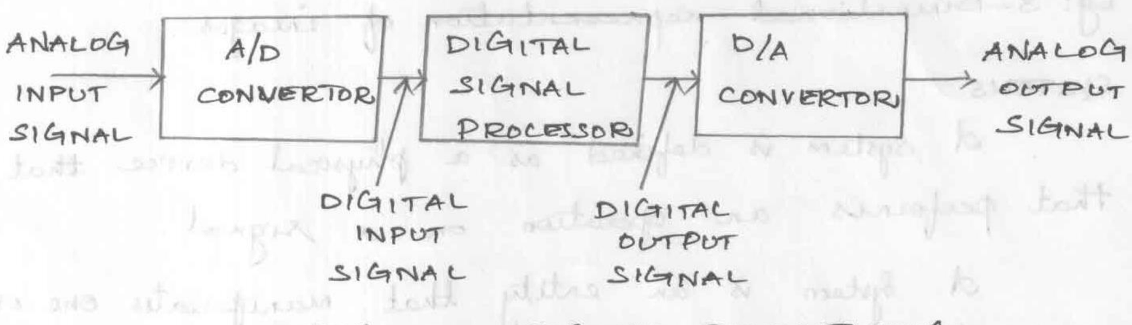
Most of the signals encountered in science and Engineering are analog in nature.

If the signal is processed directly in its analog form, then the processing is known as Analog Signal processing.



ANALOG SIGNAL PROCESSING

Eg: Filters, Amplifiers and Frequency Analyser.



DIGITAL SIGNAL PROCESSING

Processing of digital signals on a digital Computer is known as Digital Signal Processing.

To perform the processing digitally, there is a need for an interface between the analog signal and the digital processor. This interface is called as an analog to digital (A/D) convertor.

The o/p of the A/D convertor is a digital signal that is appropriate an an input to the digital processor.

The Digital Signal processor may be a digital Computer or a Small microprocessor programmed to perform the desired operations on the input signal.

Programmable machines provide the flexibility to change the signal processing operations through a change in the software, whereas hardwired machines are difficult to reconfigure.

On the other hand, when signal processing operations are well defined, a hardwired implementation which are more convenient, reliable and runs faster than its programmable counterpart.

The ~~analog to~~ digital to analog (D/A) convertor is used to provide analog ~~inform~~ signal from the digital output of the Digital Signal processor.

Example of analog o/p is speech signal (voice)

ADVANTAGES OF DIGITAL SIGNAL PROCESSING

* Flexibility in reconfiguring the digital signal processing operations simply by changing the program.

* Provides much better control of accuracy requirements.

* Digital signals are easily stored in a magnetic media without loss of signal fidelity beyond that introduced in the A/D conversion.

* Cheaper

APPLICATIONS

* Telecommunications (Modems, video conferencing, etc.
* Consumer Electronics (Digital Audio/TV, sound recor
* Instrumentation and Control. (Digital Filter, PLL)
* Image processing (Image compression, enhancement)
* Medicine (X-ray scanning, spectrum analyses of ECG and EEG signals to detect the various disorders in heart & brain)
* Speech processing
* Seismology (Detection of underground nuclear explosion & earthquake monitoring)
* Military

LIMITATIONS

* System complexity
* Bandwidth limited by sampling rate
* Power consumption

CONCEPTS OF FREQUENCY IN ANALOG AND DIGITAL SIGNALS

The concept of frequency is directly related to the concept of time.

$$f \propto \frac{1}{t}$$

Hence both for continuous and discrete time signals the nature of time(t) affects the nature of frequency(f).

ANALOG SIGNAL

A single harmonic oscillation is mathematically described by the following continuous time sinusoidal signal

$$x(t) = A\cos(\Omega t + \theta), \quad -\infty < t < \infty$$

A - Amplitude of the sinusoid

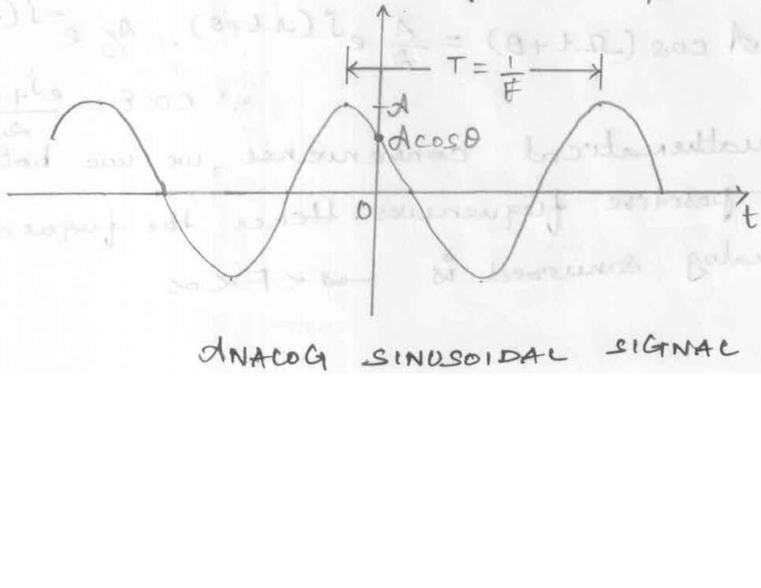
Ω - Frequency in rad/sec. $(\Omega = 2\pi F)$

θ - Phase in radians

f - Frequency in cycles per second (or) Hz $\quad \Omega = 2\pi f$

In terms of F, x(t) can be written as

$$x(t) = A\cos(2\pi Ft + \theta), \quad -\infty < t < \infty$$



ANALOG SINUSOIDAL SIGNAL

PROPERTIES

* For every fixed value of frequency F, $x(t)$ is periodic.

$$x(t+T) = x(t)$$

where, $T = \frac{1}{F}$ is the Fundamental period of the sinusoidal Signal.

* Continuous-time sinusoidal signals with distinct frequencies are themselves distinct (different)

* Increasing the Frequency F results in an increase in the rate of oscillation of the signal. i.e., the more periods are included in a given time interval.

When $F=0$, $T=\infty$ $\because F = 1/T$.

Due to continuity of the time variable t, we can increase the frequency F, without limit, with a corresponding increase in the rate of oscillation.

The sinusoidal signals carry over to the class of complex exponential can be described as

$$x(t) = A e^{j(\Omega t+\theta)}$$

$$\because x(t) = A(\cos(\Omega t+\theta) + j\sin(\Omega t+\theta))$$

$$x(t) = A\cos(\Omega t+\theta) = \frac{A}{2} e^{j(\Omega t+\theta)} + \frac{A}{2} e^{-j(\Omega t+\theta)}$$

$$\because \cos\theta = \frac{e^{j} + e^{-j}}{2}$$

For mathematical convenience, we use both negative and positive frequencies. Hence the frequency range for analog sinusoid is $-\infty < F < \infty$.

SAMPLING

Sampling is a process by which a continuous signal is converted into a sequence of discrete samples, with each sample representing the amplitude of the signal at a particular instant of time.

Sampling is described by the relation

$$x(n) = x_a(nT) \quad , \quad -\infty < n < \infty$$

$x(n)$ - discrete time signal

$x_a(t)$ - Analog signal

T - Sampling period (or) Sampling interval

Sampling rate , $F_s = 1/T$ · samples per second (or)

F_s is the sampling Frequency (hertz).

~~For a continuous time signal, the frequency~~

Periodic sampling establishes a relationship b/w the time variables t & n of continuous-time and discrete-time signals, respectively.

$$t = nT = \frac{n}{F_s}$$

As a consequence of above equation, there exist a relationship b/w the frequency of analog signal (Ω) & frequency of discrete signals.

Consider an analog sinusoidal signal,

$$x_a(t) = A\cos(2\pi F t + \theta) \longrightarrow ①$$

DIGITAL SIGNAL

A discrete-time sinusoidal signal may be expressed as

$$x(n) = A\cos(\omega n + \theta), \quad -\infty < n < \infty.$$

n - Sample number, θ - phase in radians.

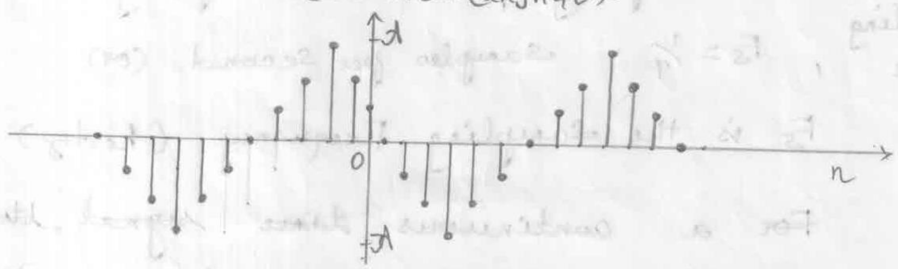
ω - frequency in radians per sample.

$\omega \equiv 2\pi f$. where $f \rightarrow$ cycles per sample.

x(n) in terms of f is expressed as

$$x(n) = A\cos(2\pi f n + \theta), \quad -\infty < n < \infty.$$

$$x(n) = A\cos(2\pi f n + \theta).$$



DISCRETE TIME SINUSOIDAL SIGNAL

PROPERTIES

* A discrete time sinusoid is periodic only if its frequency f is rational number.

$$x(n+N) = x(n) \text{ for all } n.$$

N - Fundamental Period.

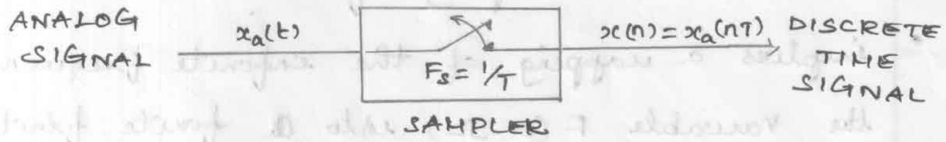
$$f = \frac{k}{N}. \qquad (\because 2\pi f_0 N = 2k\pi)$$

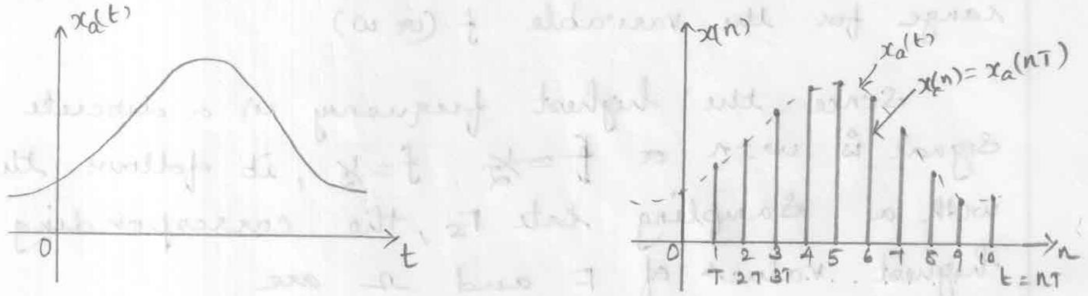
* Discrete-time sinusoids whose frequencies are separated by an integer multiple of 2π are identical

$$\cos[(\omega_0 + 2\pi)n + \theta] = \cos(\omega_0 n + 2\pi n + \theta) = \cos(\omega_0 n + \theta)$$

$$\because \cos\theta = \cos(2\pi n + \theta)$$

* The highest rate of oscillation is attained when $\omega = \pm\pi$ (or) $f = \pm\frac{1}{2}$.





PERIODIC SAMPLES OF AN ANALOG SIGNALS

which, when sampled periodically at a rate $F_s = 1/T$ samples per second, yields

$$x_a(nT) \equiv x(n) = A \cos(2\pi F nT + \theta)$$

$$x_a(nT) = A \cos\left(\frac{2\pi nF}{F_s} + \theta\right) \rightarrow ② \quad \because T = 1/F_s$$

By Comparing eqn ② with $x_n = A\cos(2\pi f n + \theta)$.

$$f = \frac{F}{F_s} \rightarrow ③ \quad (or)$$

$$\omega = \Omega T$$

Eqn ③ is known as normalized frequency.

The range of the frequency variable f for discrete time sinusoids is

$$-1/2 < f < 1/2$$
$$-\pi < \omega < \pi$$

Sub eqn ③ in the above eqn, we get

$$-\frac{1}{2T} = -\frac{F_s}{2} \le F \le \frac{F_s}{2} = \frac{1}{2T}$$

(or)

$$-\frac{\pi}{T} = -\pi F_s \le \Omega \le \pi F_s = \frac{\pi}{T}$$

Periodic sampling of a continuous-time signal implies a mapping of the infinite frequency range for the variable F (or Ω) into a finite finite frequency range for the variable f (or ω)

Since the highest frequency in a discrete time signal is ω = π or ~~f = ½~~ f = ½, it follows that, with a sampling rate F_s, the corresponding highest values of F and Ω are

$$F_{max} = \frac{F_s}{2} = \frac{1}{2T}$$

$$\Omega_{max} = \pi F_s = \frac{\pi}{T}$$

If F_M is considered to be the maximum or highest frequency in the input signal, then according to Nyquist, the sampling rate F_s should be greater than or equal to twice the maximum frequency

$$F_s \geq 2F_M.$$

When $F_s < 2F_M$, the signal cannot be reconstructed fully. Hence aliasing occurs.

This aliasing can be avoided if the input signal frequencies are below one half of the sampling frequency. Thus frequency $F_s/2$ is called the Nyquist frequency f_N (or) ω_N.

The impulse train of pulses can be expressed as

$$\delta(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s) \longrightarrow ①$$

$x_\delta(t)$ can be expressed mathematically as

$$x_\delta(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s)\, x(nT_s) \longrightarrow ②$$

$x_\delta(t)$ represent sampled version of $x(t)$.

The Fourier Transform of impulse train of eqn ① is given as

$$= f_s \sum_{n=-\infty}^{\infty} \delta(f - nf_s) \qquad \because f_s = \frac{1}{T_s}.$$

The Fourier Transform of $x_\delta(t)$ is given as

$$X_\delta(f) = f_s \sum_{n=-\infty}^{\infty} X(f - nf_s) \longrightarrow ③$$

Here $X(f)$ is the Fourier Transform of the original signal $x(t)$.

The above equation shows that a process of uniformly sampling a continuous time signal results in a periodic spectrum with period equal to f_s.

Fourier transform of signal $x(t)$ results in $X(f - nf_s)$

i.e) $X(f - nf_s) = X(f)$ at $f = 0, \pm f_s, \pm 2f_s, \pm 3f_s, \ldots$

Thus the same spectrum $X(f)$ appears at $f = 0, \pm f_s, \pm 2f_s$ etc.

Equation ③ can be written as

$$X_\delta(f) = f_s X(f) + f_s X(f \pm f_s) + f_s X(f \pm 2f_s) + f_s X(f \pm 3f_s) + \ldots \longrightarrow ④.$$

The above expression shows that every term in the sum is the same frequency spectrum at multiple sampling frequency f_s.

Eqn ③ can also be written as

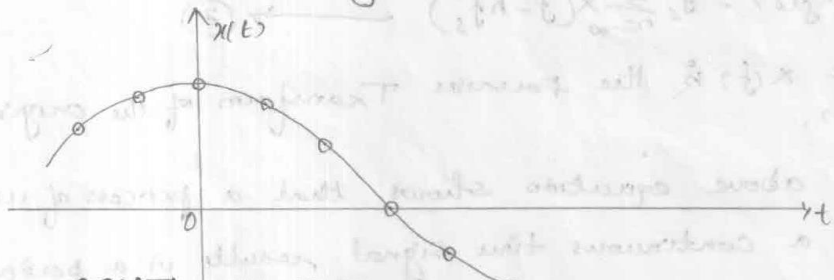
$$X_\delta(f) = f_s X(f) + \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} f_s X(f - nf_s). \longrightarrow ⑤$$

SAMPLING THEOREM

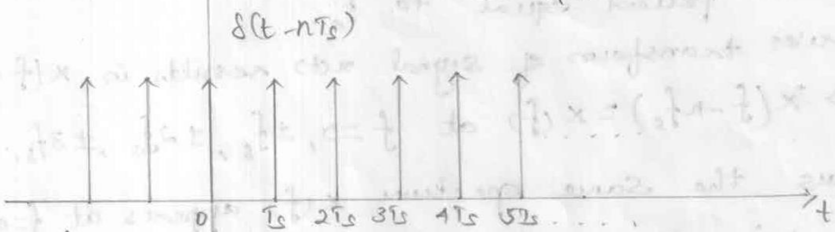
Let $x_a(t)$ is a band limited signal with $X_a(j\Omega)=$ for $|\Omega| < \Omega_m$. Then $x_a(t)$ is uniquely determined from its samples $x(n) = x_a(nT)$ if the sampling frequency $F_s \geq 2F_m$ i.e.) sampling frequency must be at least the highest frequency present in the signal.

PROOF

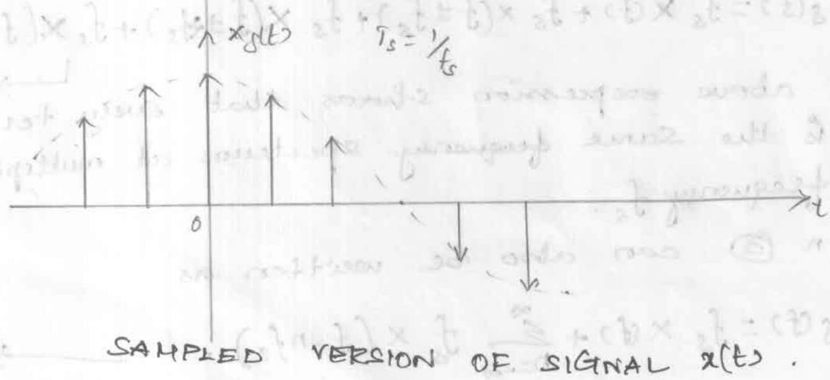
Consider $x(t)$ as a input analog signal. It has finite energy and finite duration. This $x(t)$ is band limited signal.



CONTINUOUS TIME SIGNAL $x(t)$



A UNIT IMPULSE TRAIN USED AS A SAMPLING FUNCTION



SAMPLED VERSION OF SIGNAL $x(t)$.

$$\therefore X(f) = \frac{1}{2W}\sum_{n=-\infty}^{\infty} x\left(\frac{n}{2W}\right) e^{-j2\pi f n/2W} \qquad \because T_s = \frac{1}{2W}$$

or

$$X(f) = \frac{1}{2W}\sum_{n=-\infty}^{\infty} x\left(\frac{n}{2W}\right) e^{-j\pi f n/W} \quad \text{for } -W \le f \le W \longrightarrow (6)$$

$x(t)$ is obtained from above eqn $X(f)$ by taking IFT.

$$x(t) = \text{IFT}\{X(f)\} = \text{IFT}\left\{\frac{1}{2W}\sum_{n=-\infty}^{\infty} x\left(\frac{n}{2W}\right) e^{-j\pi f n/W}\right\}.$$

or.

Sampling: $x(t) = \text{IFT}\left\{\frac{1}{f_s}\sum_{n=-\infty}^{\infty} x(nT_s)\, e^{-j2\pi f n T_s}\right\}.$

Reconstruction of signal from samples.

$$x(t) = \text{IFT}\left\{\frac{1}{2W}\sum_{n=-\infty}^{\infty} x\left(\frac{n}{2W}\right) e^{-j\pi f n/W}\right\}.$$

$$= \int_{-W}^{W} \frac{1}{2W}\sum_{n=-\infty}^{\infty} x\left(\frac{n}{2W}\right) e^{-j\pi f n/W}\, e^{j2\pi f t}\, df.$$

By Interchanging the Order of Summation & Integration

$$x(t) = \sum_{n=-\infty}^{\infty} x\left(\frac{n}{2W}\right)\frac{1}{2W}\int_{-W}^{W} e^{j2\pi f\left(t-\frac{n}{2W}\right)}\, df.$$

$$= \sum_{n=-\infty}^{\infty} x\left(\frac{n}{2W}\right)\cdot\frac{\sin(2\pi W t - n\pi)}{2\pi W t - n\pi}.$$

$$= \sum_{n=-\infty}^{\infty} x\left(\frac{n}{2W}\right)\frac{\sin\pi(2Wt-n)}{\pi(2Wt-n)}.$$

W.K.T, $\operatorname{sinc}\theta = \frac{\sin\pi\theta}{\pi\theta}.$

Reconstruction: $x(t) = \sum_{n=-\infty}^{\infty} x\left(\frac{n}{2W}\right)\operatorname{sinc}(2Wt-n) \quad -\infty < n < \infty$

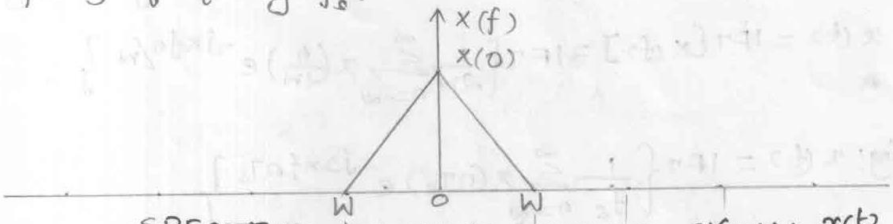
W.K.T, $T_s = \frac{1}{2W}$

$$x(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\operatorname{sinc} 2W(t-nT_s)$$

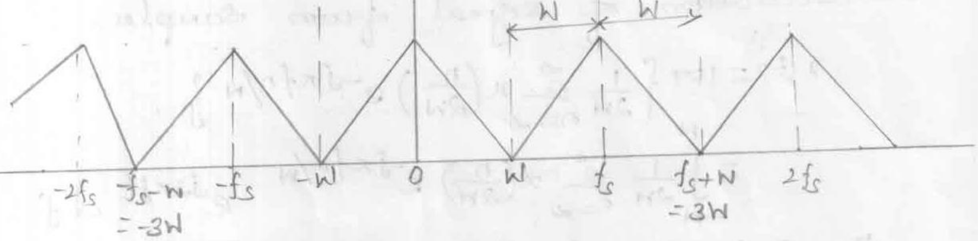
$$\therefore x(t) = x(0)\operatorname{sinc}(2Wt) + x(\pm T_s)\operatorname{sinc}_{2W}(t\pm T_s) + x(\pm 2T_s)\operatorname{sinc}_{2W}(t\pm 2T_s)$$
$$+ x(\pm 3T_s)\operatorname{sinc} 2W(t\pm 3T_s) + \dots$$

Here the amplitude of the sinc pulse changes in accordance with the sample value $x(nT_s)$.

In eqn ⑤, first term represents spectrum that would have been obtained without sampling and rest of the terms under summation represents spectrums repeating multiple frequencies of sampling frequency f_s.



SPECTRUM OF BAND LIMITED SIGNAL $x(t)$



SPECTRUM OF SAMPLED VERSION OF $x(t)$

The above spectrum is based on two assumption

1. $X(f) = 0$ for $|f| \geq W$
2. $f_s = 2W$ Sampling rate.

Eqn ⑤ can be written as

$$f_s X(f) = X_\delta(f) - \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} f_s X(f - nf_s).$$

$$\therefore X(f) = \frac{1}{f_s} X_\delta(f) - \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} X(f - nf_s)$$

$\because f_s = 2W$, above eqn will be

$$X(f) = \frac{1}{2W} X_\delta(f) - \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} X(f - nf_s)$$

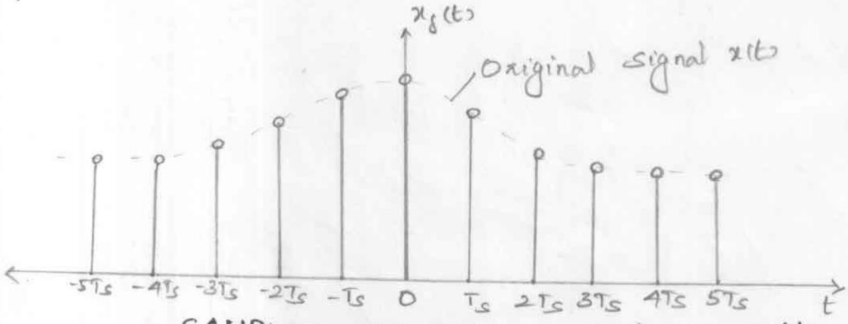
$$X(f) = \frac{1}{2W} X_\delta(f) \quad \text{for} \quad -W \leq f \leq W.$$

W.K.T, $FT[x_\delta(t)] = X_\delta(f)$.

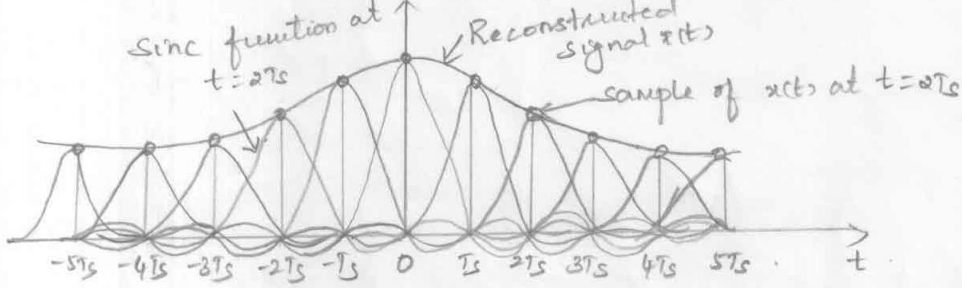
$$X(f) = \frac{1}{2W} \sum_{n=-\infty}^{\infty} x(nT_s)\, e^{-j2\pi f nT_s}.$$

where $\underbrace{X_\delta(f)}_{FT[x_\delta(t)]} = \sum_{n=-\infty} x(nT_s) e^{-j2\pi f}$

The above eqn gives F.T of $x(t)$ in terms of $x(nT_s)$



SAMPLED VERSIOF OF SIGNAL x(t)



RECONSTRUCTION OF x(t) FROM ITS SAMPLES

consider the signal $x_a(t) = 10 \cos 2\pi(1000)t + 5 \cos 2\pi(5000t)$ is to be sampled.

i) Determine the Nyquist rate for this signal.

ii) If the signal is sampled at 4 KHz, will the signal be recovered from its samples?

solution:

i) The given signal contains two cosine wave

$F_1 = 1000\,Hz$ and $A_1 = 10$

$F_2 = 5000\,Hz$ and $A_2 = 5$

$\therefore F_{Max} = 5000\,Hz$.

Nyquist rate $F_s \geq 2F_{max}$.

$F_s \geq 2(5000)$

$F_s \geq 10\,KHz$.

ii) If the signal is sampled at 4 KHz, the signal can not be recovered from its samples; because $F_s \geq 2F_{max}$.

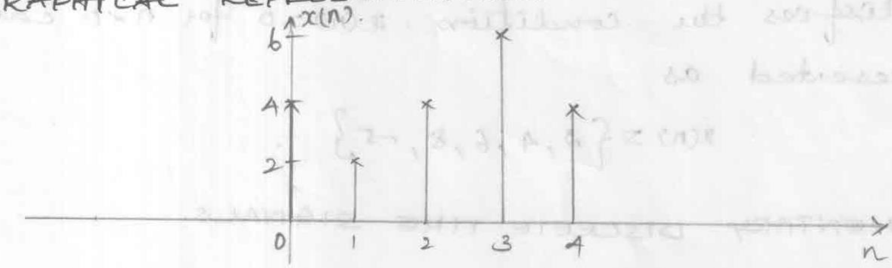
4. DISCRETE TIME SIGNALS, SYSTEMS

A continuous time signal sampled at discrete instant of time is called a discrete time signal.

The signals that are defined at discrete instants of time are known as discrete-time signal. The discrete time signals are continuous in amplitude and discrete in time.

REPRESENTATION OF DISCRETE-TIME SIGNALS

i) GRAPHICAL REPRESENTATION



$$x(n) = \{\underset{\substack{\uparrow \\ n=0}}{4}, 2, 4, 6, 4\}$$

ii) FUNCTIONAL REPRESENTATION

$$x(n) = \begin{cases} 1 & \text{for } n = -1 \\ 2 & \text{for } n = 0, 1 \\ 0.5 & \text{for } n = 2 \\ 3.5 & \text{for } n = 3 \\ 0 & \text{Otherwise.} \end{cases}$$

iii) TABULAR REPRESENTATION

n	-1	0	1	2	3
x(n)	1	2	5	0.5	1.5

iv) SEQUENCE REPRESENTATION

A finite duration sequence with time origin (n=0) indicated by the symbol ↑ is represented as

$$x(n) = \{1, \underset{\uparrow}{2}, 2, 0.5, 1.5\}$$

An infinite duration sequence can be represented as

$$x(n) = \{\ldots, 0.5, \underset{\uparrow}{1}, -2, 3, 5, \ldots\}$$

A finite duration sequence that satisfies the condition $x(n) = 0$ for $n<0$ can be represented as

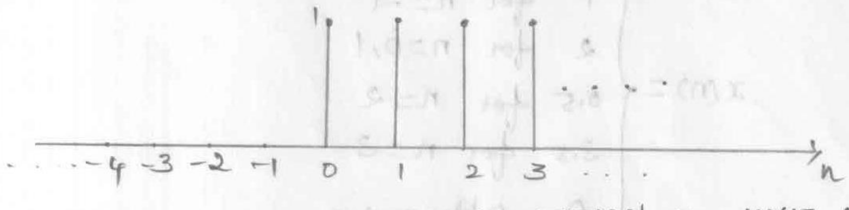
$$x(n) = \{2, 4, 6, 8, -5\}.$$

ELEMENTARY DISCRETE TIME SIGNALS

i) UNIT STEP SIGNALS

The unit step sequence is defined as

$$u(n) = 1 \text{ for } n \geq 0$$
$$= 0 \text{ for } n < 0$$

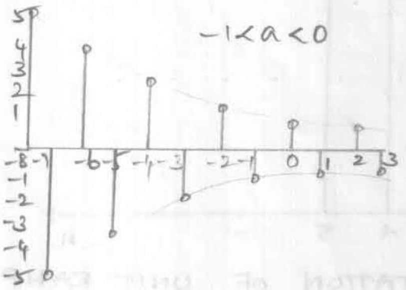


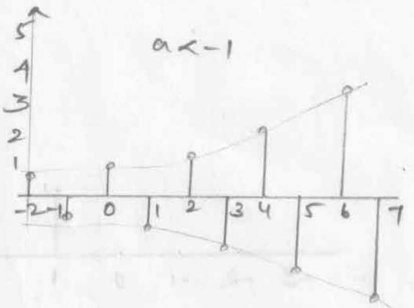
GRAPHICAL REPRESENTATION OF UNIT STEP SIGNAL

ii) UNIT RAMP SEQUENCE

The unit ramp sequence is defined as

$$r(n) = n \text{ for } n \geq 0$$
$$= 0 \text{ for } n < 0$$



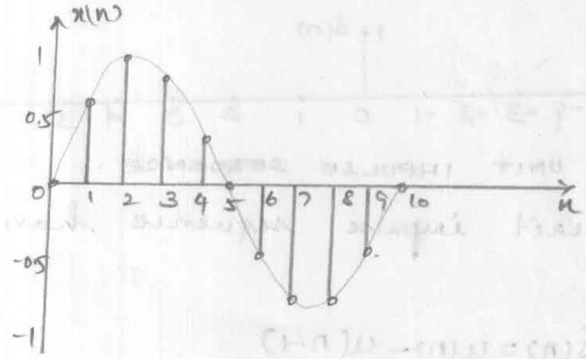


EXPONENTIAL SEQUENCE

V) SINUSOIDAL SIGNAL

The discrete-time sinusoidal signal is given by

$$x(n) = A\cos(\omega_0 n + \phi)$$



A SINUSOIDAL SIGNAL

VI) COMPLEX EXPONENTIAL SIGNAL

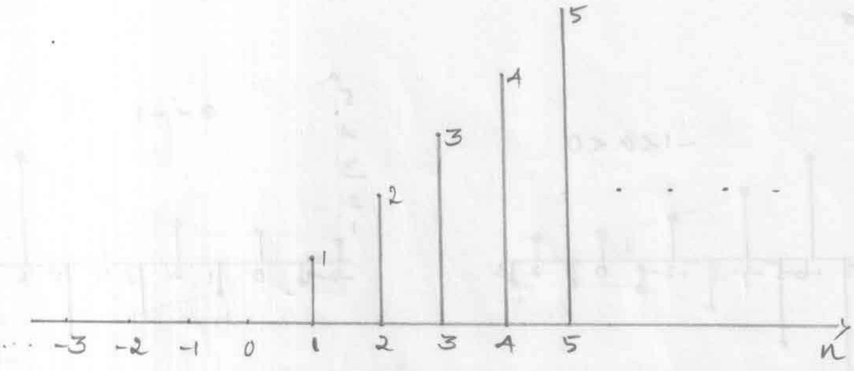
The discrete time Complex exponential signal is given by

$$x(n) = a^n e^{j(\omega_0 n + \phi)}$$

$$= a^n \cos(\omega_0 n + \phi) + j a^n \sin(\omega_0 n + \phi)$$

For $|a| = 1$, the real and imaginary part of Complex exponential sequence are sinusoidal.

For $|a| < 1$, the amplitude of the sinusoidal sequence decays exponentially and for $|a| > 1$, the amplitude of the sinusoidal sequence increases exponentially

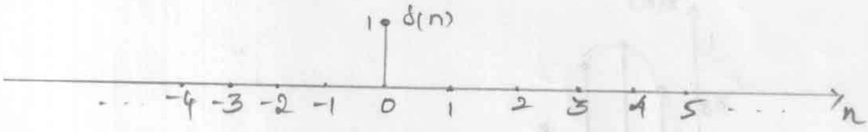


GRAPHICAL REPRESENTATION OF UNIT RAMP SEQUENCE

iii) UNIT IMPULSE SEQUENCE

The unit impulse sequence is defined as

$$\delta(n) = 1 \text{ for } n = 0$$
$$= 0 \text{ for } n \neq 0$$



UNIT IMPULSE SEQUENCE

iv) The unit impulse sequence has the following properties

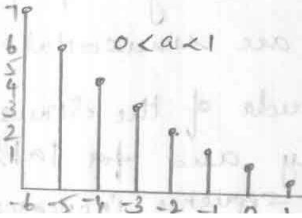
$$\delta(n) = u(n) - u(n-1)$$

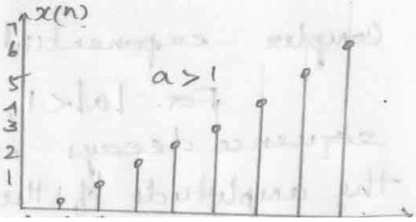
$$u(n) = \sum_{k=\infty}^{\infty} \delta(k)$$

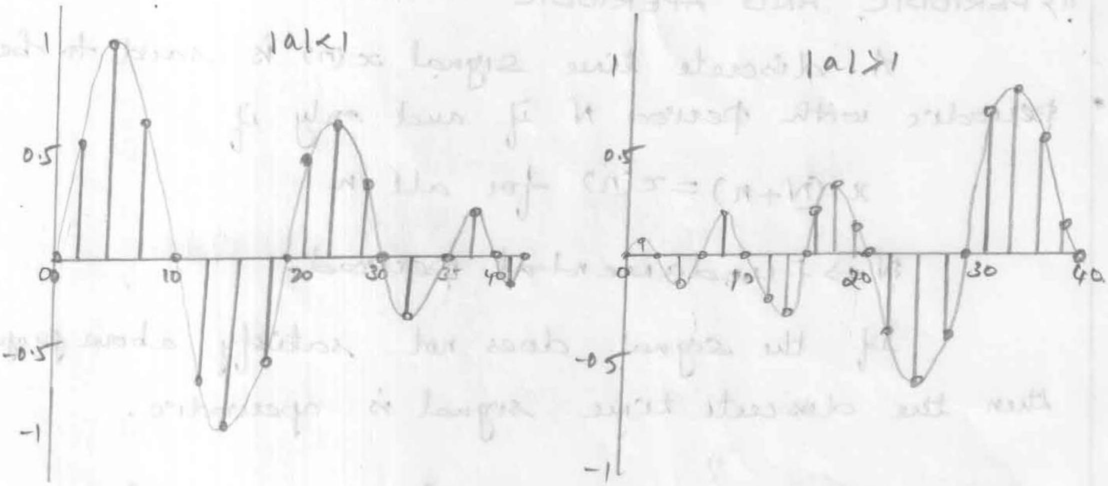
$$\sum_{n=-\infty}^{\infty} x(n)\,\delta(n-n_0) = x(n_0)$$

iv) EXPONENTIAL SEQUENCE

The exponential signal is a sequence of the form $x(n) = a^n$ for all n.







COMPLEX EXPONENTIAL SEQUENCE

CLASSIFICATION OF DISCRETE-TIME SIGNALS.

i) ENERGY AND POWER SIGNALS

For descrete-time signal $x(n)$ the energy E is defined as

$$E = \sum_{n=-\infty}^{\infty} |x(n)|^2$$

The average power of a descrete time signal $x(n)$ is defined as

$$P = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x(n)|^2$$

A signal is an energy signal, if and only if the total energy of the signal is finite, for an energy signal $P=0$

The signal is said to be power signal if the average power of the signal is finite.

For a power signal $E=\infty$

~~That~~ The signal that are not satisfy above properties are ~~naught~~ neither energy nor power signals.

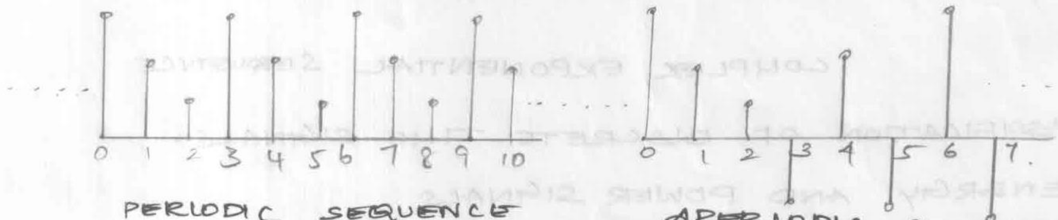
ii) PERIODIC AND APERIODIC SIGNALS

A discrete time signal $x(n)$ is said to be periodic with period N if and only if

$$x(N+n) = x(n) \quad \text{for all } n$$

N → Fundamental period.

If the signal does not satisfy above property, then the discrete time signal is aperiodic.



PERIODIC SEQUENCE

APERIODIC SEQUENCE.

iii) SYMMETRIC (EVEN) AND ANTISYMMETRIC (ODD) SIGNALS

A discrete time signal $x(n)$ is said to be a symmetric (even) signal if it satisfies the condition

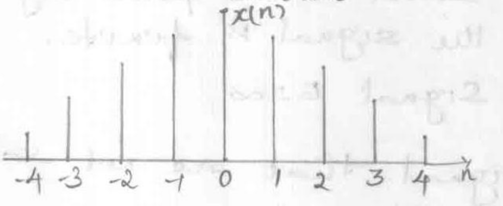
$$x(-n) = x(n) \quad \text{for all } n$$

Eg: $x(n) = \cos \omega n$

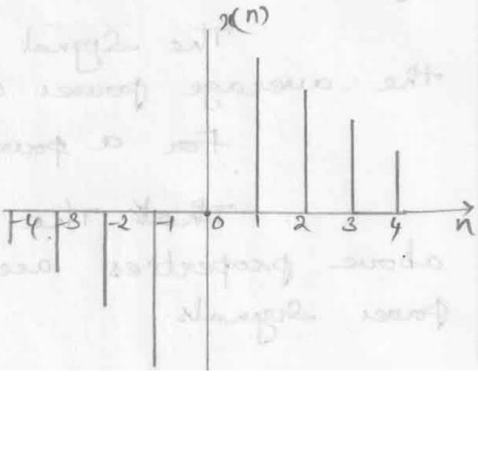
The signal is said to be an odd signal if it satisfies the condition

$$x(-n) = -x(n) \quad \text{for all } n$$

Eg: $x(n) = \sin \omega n$



A SYMMETRIC SIGNAL



iv) CAUSAL AND NONCAUSAL SIGNALS

A signal is said to be causal if its value is zero for $n<0$. Otherwise the signal is noncausal.

Eg: Causal Signal

$$x_1(n) = a^n u(n)$$

$$x_2(n) = \{\underset{\uparrow}{1}, 2, -3, -1, 2\}$$

Non causal signal

$$x_1(n) = a^n u(-n+1)$$

$$x_2(n) = \{1, -2, \underset{\uparrow}{1}, 4, 5\}$$

A signal that is zero for all $n \geq 0$ is called an anticausal signal.

OPERATIONS ON SIGNALS

The mathematical transformation from one signal to another is represented as

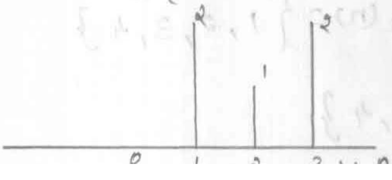
$$y(n) = T[x(n)].$$

i) SHIFTING

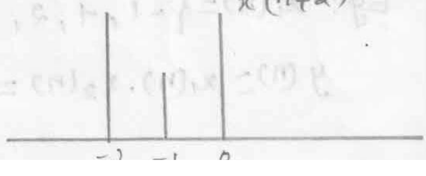
$$y(n) = x(n-k)$$

If k is positive, the shifting delays the sequence.
If k is negative, the shifting advances the sequence.

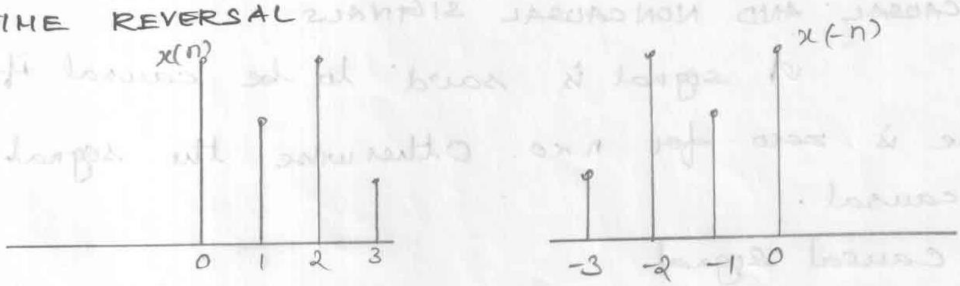
$x(n-1)$



$x(n+2)$

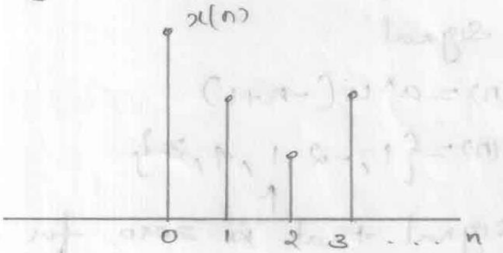


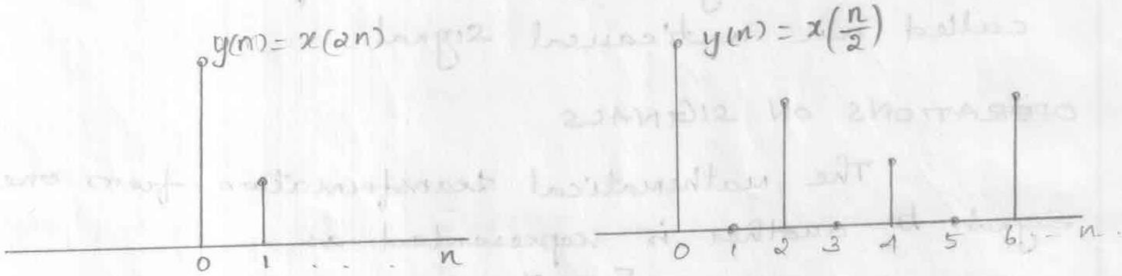
ii) TIME REVERSAL



iii) TIME SCALING

$y(n) = x(2n)$



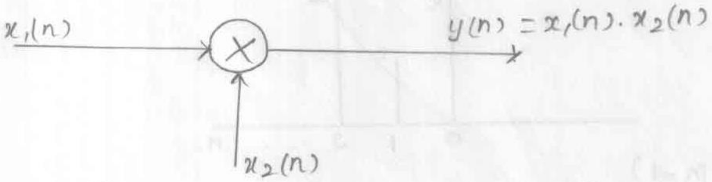


GRAPHICAL ILLUSTRATION OF TIME SCALING

iv) SCALAR MULTIPLICATION

Eg: $x(n) = \{1, 2, 3\}$, $a = 2$; $y(n) = \{2, 4, 6\}$.

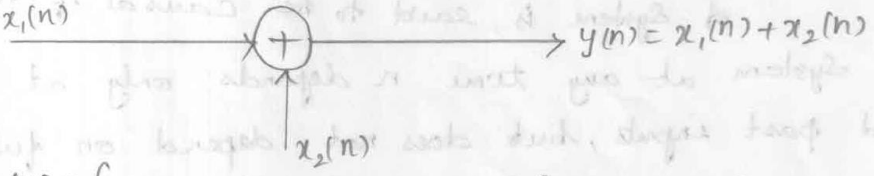
v) SIGNAL MULTIPLIER



Eg: $x_1(n) = \{-1, 4, 2, 1\}$ $\quad x_2(n) = \{1, 2, 3, 4\}$

$y(n) = x_1(n) \cdot x_2(n) = \{-1, 8, 6, 4\}$

vi) ADDITION OPERATION



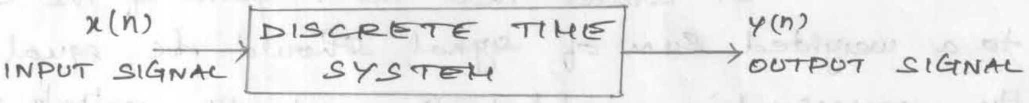
Eg: $x_1(n) = \{1, 3, 2, 5\}$ $x_2(n) = \{3, 1, 2, 4\}$

$y(n) = x_1(n) + x_2(n) = \{4, 4, 4, 9\}$

DISCRETE TIME SYSTEMS

A discrete-time system is a device or an algorithm that operates on a discrete-time input signal $x(n)$, according to some well defined rule, to produce another discrete-time signal $y(n)$ called the output signal.

$$y(n) = T[x(n)]$$



CLASSIFICATION OF DISCRETE TIME SYSTEMS

i) STATIC AND DYNAMIC SYSTEMS

If Discrete time system is static system, that output depends at any instant n, depends on the i/p samples at the same time, but not depends on past a future samples of the input.

Eg: Static System

$y(n) = a\,x(n)$

$y(n) = a\,x^2(n)$

Dynamic System

$y(n) = x(n-1) + x(n-2)$

$y(n) = x(n+1) + x(n)$

ii) CAUSAL AND NONCAUSAL SYSTEMS

A System is said to be causal if the o/p of the System at any time n depends only at present and past inputs, but does not depend on future inputs.

$$y(n) = F[x(n), x(n-1), x(n-2), \ldots]$$

Eg: causal System

$$y(n) = x(n) + x(n-1)$$

Non-causal System

$$y(n) = x(2n)$$

iii) LINEAR AND NON-LINEAR SYSTEMS

A System that satisfies the superposition principle is said to be a Linear System.

superposition principle:

It states that the response of the System to a weighted sum of signal should be equal to the corresponding weighted sum of the outputs of the System to each of the individual input signal.

$$T[a_1 x_1(n) + a_2 x_2(n)] = a_1 T[x_1(n)] + a_2 T[x_2(n)]$$

iv) TIME VARIENT AND TIME INVARIENT SYSTEMS

A System is said to be time-invarient or shift invarient if the characteristics of the System do not change with time.

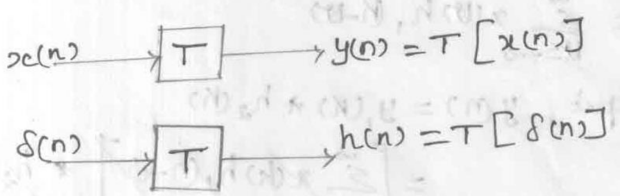
$$y(n) \longleftrightarrow T[x(n)]$$

$$y(n-k) \longleftrightarrow T[x(n-k)]$$

ANALYSIS OF DISCRETE TIME LTI SYSTEMS

A linear time-invariant (LTI) discrete-time system satisfies both the linearity and the time invariance properties.

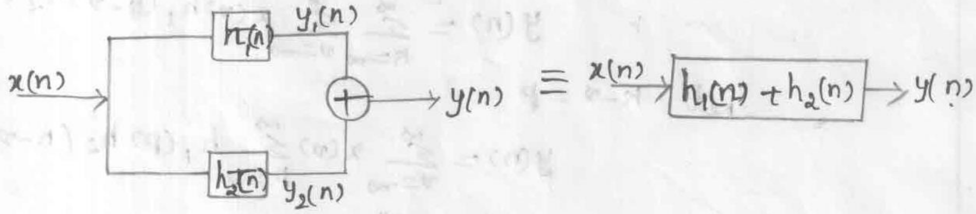
When the input to a system is an unit impulse signal, the response of the system is called an impulse response.



$h(n)$ – impulse response.

INTERCONNECTION OF LTI SYSTEMS

i) PARALLEL CONNECTION OF SYSTEMS



$$y_1(n) = x(n) * h_1(n)$$

$$y_2(n) = x(n) * h_2(n)$$

$$y(n) = y_1(n) + y_2(n)$$

$$= x(n) * h_1(n) + x(n) * h_2(n)$$

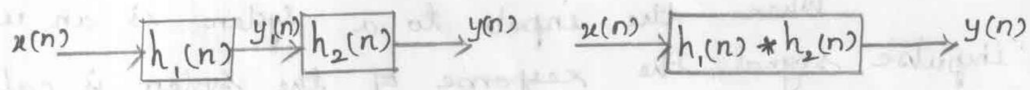
$$= \sum_{k=-\infty}^{\infty} x(k)\, h_1(n-k) + \sum_{k=-\infty}^{\infty} x(k)\, h_2(n-k)$$

$$= \sum_{k=-\infty}^{\infty} x(k)\left[h_1(n-k) + h_2(n-k)\right]$$

$$= \sum_{k=-\infty}^{\infty} x(k)\, h(n-k) \qquad \because h(n) = h_1(n) + h_2(n)$$

If the two systems are connected in parallel, then the overall impulse response is equal to sum of two impulse responses.

ii) CASCADE CONNECTION OF TWO SYSTEMS



$$y_1(k) = x(k) * h_1(k)$$

$$= \sum_{k=-\infty}^{\infty} x(k)\, h_1(k-k)$$

The output, $y(n) = y_1(k) * h_2(k)$

$$= \left[\sum_{k=-\infty}^{\infty} x(k)\, h_1(n-k)\right] * h_2(n)$$

$$= \sum_{k=-\infty}^{\infty} \sum$$

$$= \left[\sum_{v=-\infty}^{\infty} x(v)\, h_1(k-v)\right] * h_2(k)$$

$$y(n) = \sum_{k=-\infty}^{\infty} \sum_{v=-\infty}^{\infty} x(v)\, h_1(k-v)\, h_2(n-k)$$

Let $k - v = p$

$$y(n) = \sum_{v=-\infty}^{\infty} x(v) \sum_{p=-\infty}^{\infty} h_1(p)\, h_2(n-v+p)$$

$$= \sum_{v=-\infty}^{\infty} x(v)\, h(n-v)$$

$$= x(n) * h(n)$$

where $h(n) = \sum_{k=-\infty}^{\infty} h_1(k)\, h_2(n-k)$

$$= h_1(n) * h_2(n).$$

Hence the impulse response of two LTI systems connected in cascade is the convolution of the individual impulse response.

1) Determine the values of power and energy of the following signals. Find whether the signals are power or energy signal.

i) $x(n) = \left(\frac{1}{3}\right)^n u(n)$ ii) $x(n) = e^{j\left[\frac{\pi}{2}n + \frac{\pi}{4}\right]}$ iii) $x(n) = \sin\left(\frac{\pi}{4}n\right)$

Solution:

i) Given $x(n) = \left(\frac{1}{3}\right)^n u(n)$

The energy of the signal

$$E = \sum_{n=-\infty}^{\infty} |x(n)|^2$$

$$= \sum_{n=0}^{\infty} \left[\left(\tfrac{1}{3}\right)^n\right]^2 \qquad \because u(n) = 1 \text{ for } n \geq 0, \; = 0 \text{ for } n < 0$$

$$= \sum_{n=0}^{\infty} \left(\tfrac{1}{9}\right)^n$$

$$= \frac{1}{1 - \frac{1}{9}} = \frac{9}{8}.$$

The power $$P = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=0}^{N} \left(\tfrac{1}{9}\right)^n$$

$$= \lim_{N \to \infty} \frac{1}{2N+1} \left[\frac{1 - \left(\frac{1}{9}\right)^{N+1}}{1 - \frac{1}{9}}\right]$$

$$= 0$$

The energy is finite and power is zero.

$\therefore$ the signal is an energy signal.

ii) $x(n) = e^{j\left(\frac{\pi}{2}n + \frac{\pi}{4}\right)}$

$$E = \sum_{n=-\infty}^{\infty} \left| e^{j\left(\frac{\pi}{2}n + \frac{\pi}{4}\right)} \right|^2 \qquad \because |e^{j(\omega + \theta)}| = 1.$$

$$= \sum_{n=-\infty}^{\infty} 1 = \infty$$

$$P = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x(n)|^2$$

$$= \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} \left| e^{j\left(\frac{\pi}{2}n + \frac{\pi}{4}\right)} \right|^2$$

$$= \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} 1 \qquad \because \sum_{n=-N}^{N} 1 = 2N+1.$$

$$= 1.$$

iii) $x(n) = \sin(\pi/4\, n)$

$$E = \sum_{n=-\infty}^{\infty} |\sin^2(\pi/4\, n)| = \sum_{n=-\infty}^{\infty}\left[\frac{1-\cos(\pi/2\, n)}{2}\right] = \infty$$

$$P = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |\sin^2(\pi/4\, n)|$$

$$= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} \frac{1-\cos \pi/2\, n}{2} = \frac{1}{2} \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} 1$$

$$= \frac{1}{2}.$$

This signal is power signal.

Determine whether or not each of the following signals is periodic. If a signal is periodic, specify its fundamental period.

i) $x(n) = e^{j6\pi n}$ ii) $x(n) = e^{j\,3/5(n+1/2)}$ iii) $x(n) = \cos(2\pi/3)\, n$

iv) $x(n) = \sin\left(\frac{\pi n}{4}\right)$ v) $x(n) = \cos \pi/3\, n + \cos 3\pi/4\, n$.

SOLUTION:

i) $x(n) = e^{j6\pi n}$. It is in the form of $x(n) = e^{j\omega_0 n}$.

$\omega_0 = 6\pi$.

The fundamental frequency is multiple of π.

$\therefore$ The signal is periodic.

$$N = 2\pi\left(\frac{m}{\omega_0}\right) = 2\pi\left(\frac{m}{6\pi}\right)$$

The minimum value of m for which N is integer is 3.

$$N = 2\pi\left(3/6\pi\right) = 1.$$

$\therefore$ The fundamental period = 1.

ii) $x(n) = e^{j\,3/5(n+1/2)}$

$\omega_0 = 3/5$ which is not a multiple of π.

$\therefore$ The signal is aperiodic.

iii) $x(n) = \cos(2\pi/3)\, n$, $\omega_0 = 2\pi/3$ $\therefore$ The signal is periodic.

The fundamental period

$N = 2\pi\left(\frac{m}{2\pi/3}\right) = 3m$ for $m=1$.

$N = 3.$

iv) $x(n) = \sin(\pi n/4)$.

$\omega_0 = \pi/4$.

This is periodic signal.

$N = 2\pi\left(\frac{m}{\omega_0}\right) = 2\pi\left(\frac{m}{\pi/4}\right) = 8m$.

$N = 8$ for $m = 1$.

v) $x(n) = \cos(\pi/3\, n) + \cos(3\pi/4)\, n$

the fundamental period of the signal $\cos(\pi/3\, n)$ is

$N_1 = 2\pi\left(\frac{m}{\pi/3}\right) = 6$. for $m = 1$.

|||ly $N_2 = 2\pi\left(\frac{m}{3\pi/4}\right) = 8$ for $m = 3$.

$\frac{N_1}{N_2} = 6/8 = 3/4$.

$N = 4N_1 = 3N_2 = 24$.

3) Find whether the following systems are static or dynamic.

i) $y(n) = x(n)\, x(n-1)$ ii) $y(n) = x^2(n) + x(n)$

SOLUTION:

i) $y(n) = x(n)\, x(n-1)$

The output $y(n)$ depends on the past input.

∴ The system is dynamic.

ii) $y(n) = x^2(n) + x(n)$

The output $y(n)$ depends on the input at that instant only.

∴ The system is static.

4) Determine if the systems described by the following equations are causal or non-causal

i) $y(n) = x(n) + \frac{1}{x(n-1)}$ ii) $y(n) = x(n^2)$

Solution:

i) $y(n) = x(n) + \frac{1}{x(n-1)}$

For $n=-1$; $y(-1) = x(-1) + \frac{1}{x(-2)}$

For $n=0$; $y(0) = x(0) + \frac{1}{x(-1)}$

For $n=1$; $y(1) = x(1) + \frac{1}{x(0)}$

For all the values of n, the output depends on present and past inputs.

∴ The system is causal.

ii) $y(n) = x(n^2)$

For $n=-1$, $y(-1) = x(1)$

For $n=0$, $y(0) = x(0)$

For $n=1$, $y(1) = x(1)$

For all values of n, the system depends on future inputs. So, the system is non-causal.

5) Determine if the system described by the following input-output equations is linear or non linear.

i) $y(n) = x(n) + \frac{1}{x(n-1)}$ ii) $y(n) = x^2(n)$ iii) $y(n) = n\,x(n)$

Solution:

i) Given $y(n) = x(n) + \frac{1}{x(n-1)}$

For two i/p sequences $x_1(n)$ & $x_2(n)$ the corresponding o/p's are

$y_1(n) = T[x_1(n)] = x_1(n) + \frac{1}{x_1(n-1)}$

$y_2(n) = T[x_2(n)] = x_2(n) + \frac{1}{x_2(n-1)}$

The o/p due to weighted sum of inputs is

$y_3(n) = T[a_1 x_1(n) + a_2 x_2(n)]$

$y_3(n) = a_1 x_1(n) + a_2 x_2(n) + \frac{1}{a_1 x_1(n-1) + a_2 x_2(n-1)}$ → (1)

The linear combination of the two outputs is

$$a_1 y_1(n) + a_2 y_2(n) = a_1 x_1(n) + \frac{a_1}{x_1(n-1)} + a_2 x_2(n) + \frac{a_2}{x_2(n-1)} \longrightarrow (2)$$

equ's (1) ≠ (2).

∴ The system is non-linear.

ii) $y(n) = x^2(n)$.

$y_1(n) = T[x_1(n)] = x_1^2(n)$

$y_2(n) = T[x_2(n)] = x_2^2(n)$.

$$y_3(n) = T[a_1 x_1(n) + a_2 x_2(n)] = [a_1 x_1(n) + a_2 x_2(n)]^2 \longrightarrow (1)$$

$$a_1 T[x_1(n)] + a_2 T[x_2(n)] = a_1 x_1^2(n) + a_2 x_2^2(n) \longrightarrow (2).$$

Eqn's (1) ≠ (2)

∴ The system is non-linear.

iii) $y(n) = n\,x(n)$.

$y_1(n) = T[x_1(n)] = n\,x_1(n)$

$y_2(n) = T[x_2(n)] = n\,x_2(n)$.

$$y_3(n) = T[a_1 x_1(n) + a_2 x_2(n)] = n a_1 x_1(n) + n a_2 x_2(n) \longrightarrow (1).$$

$$a_1 T[x_1(n)] + a_2 T[x_2(n)] = a_1 n x_1(n) + a_2 n x_2(n) \longrightarrow (2).$$

(1) = (2), ∴ The system is linear.

6) Determine if the following systems are time-invariant or time variant.

i) $y(n) = x(n) + x(n-1)$ ii) $y(n) = x(-n)$

Solution:

i) Given $y(n) = x(n) + x(n-1)$.

If the input is delayed by k units in time, we have

$y(n,k) = T[x(n-k)] = x(n-k) + x(n-k-1)$

If we delay the o/p by k units in time then

$y(n-k) = x(n-k) + x(n-k-1)$

UNIT- 2

Z TRANSFORM

The z-transform is a powerful tool for the analysis of linear-time invariant discrete time systems in the frequency domain.

In z-domain the convolution of two time domain signals is equivalent to multiplication of their corresponding z-transform.

DEFINITION

The z-transform of a discrete-time signal $x(n)$ is defined as

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} \quad \text{(Two sided z-transform)}.$$

Where z is a complex variable

In polar form z can be expressed as

$$z = re^{j\omega}$$

Where r is the radius of the circle.

$$X(re^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n) r^{-n} e^{-j\omega n}$$

For $r=1$, the above expression reduces to Fourier transform of $x(n)$.

i.e. the z-transform evaluated on the unit circle corresponds to the Fourier transform.

If $x(n)$ is a causal sequence i.e. $x(n)=0$ for $n<0$, then the z-transform is

$$X_+(z) = \sum_{n=0}^{\infty} x(n) z^{-n} \quad \text{(One sided z-transform)}$$

PROPERTIES OF THE Z-TRANSFORM

i) LINEARITY

If $X_1(z) = Z\{x_1(n)\}$ and $X_2(z) = Z\{x_2(n)\}$, then

$$Z\{a\,x_1(n) + b\,x_2(n)\} = a\,X_1(z) + b\,X_2(z)$$

PROOF:

$$Z\{a\,x_1(n) + b\,x_2(n)\} = \sum_{n=-\infty}^{\infty} [a\,x_1(n) + b\,x_2(n)]\, z^{-n}$$

$$= a \sum_{n=-\infty}^{\infty} x_1(n)\, z^{-n} + b \sum_{n=-\infty}^{\infty} x_2(n)\, z^{-n}$$

$$= a\,X_1(z) + b\,X_2(z)$$

ii) TIME SHIFT OR TRANSLATION

If $X(z) = Z\{x(n)\}$ and the initial conditions for $x(n)$ are zeros, then

$$Z\{x(n-m)\} = z^{-m}\, X(z)$$

PROOF:

$$Z\{x(n-m)\} = \sum_{n=-\infty}^{\infty} x(n-m)\, z^{-n}$$

$$= z^{-m} \sum_{n=-\infty}^{\infty} x(n-m)\, z^{-(n-m)}$$

Let $(n-m) = l$, then

$$Z\{x(n-m)\} = z^{-m} \sum_{l=-\infty}^{\infty} x(l)\, z^{-l}$$

$$Z\{x(n-m)\} = z^{-m}\, X(z)$$

iii) MULTIPLICATION BY AN EXPONENTIAL SEQUENCE

If $X(z) = Z\{x(n)\}$, then

$$Z\{a^n x(n)\} = X(a^{-1}z)$$

PROOF:

$$Z\{a^n x(n)\} = \sum_{n=-\infty}^{\infty} a^n x(n)\, z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} x(n)\, (a^{-1}z)^{-n} = X(a^{-1}z)$$

iv) TIME REVERSAL

If $X(z) = Z\{x(n)\}$, then

$$Z\{x(-n)\} = X(z^{-1})$$

PROOF:

$$Z\{x(-n)\} = \sum_{n=-\infty}^{\infty} x(-n) z^{-n}$$

$$= \sum_{l=-\infty}^{\infty} x(l) (z^{-1})^{-l} \quad \text{where } l = -n$$

$$= \sum_{l=-\infty}^{\infty} = X(z^{-1})$$

where the ROC is $\frac{1}{R_2} < |z| < \frac{1}{R_1}$

v) DIFFERENTIATION OF X(z)

If $X(z) = Z\{x(n)\}$, then

$$Z\{n\,x(n)\} = -z \frac{d}{dz} X(z)$$

PROOF:

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

Differentiating the z-transform, we get

$$\frac{d}{dz} X(z) = \sum_{n=-\infty}^{\infty} x(n) \frac{d}{dz} z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} x(n) (-n) z^{-n-1}$$

$$= -\frac{1}{z} \sum_{n=-\infty}^{\infty} n x(n) z^{-n}$$

$$-z \frac{d}{dz} X(z) = \sum_{n=-\infty}^{\infty} n\,x(n)\, z^{-n}$$

$$= Z\{n\,x(n)\}$$

vi) CONVOLUTION THEOREM

If $X(z) = Z\{x(n)\}$, and $H(z) = Z\{h(n)\}$, then

$$Z\{x(n) * h(n)\} = X(z) H(z)$$

$x(n) * h(n) \rightarrow$ Linear Convolution Sequences.

PROOF:

We have $y(n) = x(n) * h(n)$

$$= \sum_{k=-\infty}^{\infty} x(k) h(n-k) \quad \text{and}$$

$$Y(z) = Z\{y(n)\} = \sum_{n=-\infty}^{\infty} \left[\sum_{k=-\infty}^{\infty} x(k) h(n-k) \right] z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} x(k) z^{-k} h(n-k) z^{-(n-k)}$$

Interchange the order of the summation

$$Y(z) = \sum_{k=-\infty}^{\infty} x(k) z^{-k} \sum_{n=-\infty}^{\infty} h(n-k) z^{-(n-k)}$$

$$= \sum_{k=-\infty}^{\infty} x(k) z^{-k} \sum_{l=-\infty}^{\infty} h(l) z^{-l} \quad \text{where } l = n-k$$

$$Z\{x(n) * h(n)\} = X(z) H(z)$$

vii) PARSEVAL'S RELATION

Let us consider two complex sequences $x_1(n)$ and $x_2(n)$. Parseval's relation states that

$$\sum_{n=-\infty}^{\infty} x_1(n) x_2^*(n) = \frac{1}{2\pi j} \oint_C X_1(v) X_2^*\left(\frac{1}{v^*}\right) v^{-1} dv$$

Where the contour of integration must be in the overlap of the regions of convergence of $X_1(v)$ and $X_2^*\left(\frac{1}{v^*}\right)$

PROOF:

Let $y(n) = x_1(n)\, x_2^*(n)$. Then $Y(z) = \sum_{n=-\infty}^{\infty} x_1(n)\, x_2^*(n)\, z^{-n}$

viii) PARSEVAL'S RELATION

Parseval's relation states that the total average power in a discrete periodic signal $x(n)$, equals the sum of the average power, in individuals harmonic components, which in turn equals to squared magnitude of $x(z)$.

$$\sum_{n=-\infty}^{\infty} |x(n)|^2 \xleftrightarrow{ZT} \frac{1}{2\pi j} \oint_C |X(z)|^2$$

PROOF

If $x(n)$ is the discrete periodic signal, then the average power of a periodic signal is given by

$$\sum_{n=-\infty}^{\infty} |x(n)|^2 = \sum_{n=-\infty}^{\infty} x(n) \left[x(n)\right]^*$$

$$= \sum_{n=-\infty}^{\infty} x(n) \left[\frac{1}{2\pi j} \oint_C |X(z)\, z^n|\right]^*$$

Interchanging the position of integral and summation,

$$\sum_{n=-\infty}^{\infty} |x(n)|^2 = \frac{1}{2\pi j} \oint_C X^*(z) \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$$

$$= \frac{1}{2\pi j} \oint_C X^*(z)\, X(z)$$

$$= \frac{1}{2\pi j} \oint |X(z)|^2$$

$$\therefore \sum_{n=-\infty}^{\infty} |x(n)|^2 \xleftrightarrow{ZT} \frac{1}{2\pi j} \oint_C |X(e^{j\omega})|^2.$$

viiii) INITIAL VALUE THEOREM

If $x(n) = 0,\ n < 0$, then

$$x(0) = \lim_{z \to \infty} X(z)$$

ix) FINAL VALUE THEOREM

If $x(n) \overset{ZT}{\longleftrightarrow} X(Z)$, then

$$x(\infty) = \underset{Z \to 1}{Lt} (Z-1) X(Z)$$

x) CORRELATION

If $X_1(Z) = Z\{x_1(n)\}$ and $X_2(Z) = Z\{x_2(n)\}$ then

$$Z\left[r_{x_1 x_2}(l)\right] = Z\left[\sum_{n=-\infty}^{\infty} x_1(n)\, x_2(n-l)\right]$$

$$= \Gamma_{x_1 x_2}(Z) = X_1(Z)\, X_2(Z^{-1})$$

Z-TRANSFORM AND ROC OF FINITE DURATION SEQUENCES.

The region of convergence (ROC) of $X(z)$ is the set of all values of z for which $X(z)$ attains a finite value.

RIGHT HAND SEQUENCE

A right hand ~~side~~ sequence is one for which $x(n) = 0$ for all $n < n_0$ where n_0 is +ve or -ve but finite.

If n_0 is greater than or equal to zero, the resulting sequence is causal or a positive time sequence.

For such type of sequence the ROC is entire z plane except $z=0$.

Eg: 1

Find the z-transform and ROC of the causal sequence

$$x(n) = \{\underset{\uparrow}{1}, 0, 3, -1, 2\}$$

Solution:

For causal sequence res $x(n) = 0$ for $n<0$, then the z transform is

$$X_+(z) = \sum_{n=0}^{\infty} x(n) z^{-n}$$

$$= x(0) z^0 + x(1) z^{-1} + x(2) z^{-2} + x(3) z^{-3} + x(4) z^{-4}.$$

$$X_+(z) = 1 + 3z^{-2} - z^{-3} + 2z^{-4}$$

The $X_+(z)$ converges for all values of z except at $z=0$.

LEFT HAND SEQUENCE

$$x_n = 0 \quad \text{for all } n \geq n_0$$

If $n_0 < 0$ the resulting sequence is anticausal sequence.

For such type of sequence the ROC is entire z-plane except at $z = \infty$

Eg: 2

Find the z-transform and ROC of the anticausal sequence.

$$x(n) = \{-3, -2, -1, 0, \underset{\uparrow}{1}\}$$

Solution.

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$X(z) = 1 - z^{2} - 2z^{3} - 3z^{4}.$$

The $X(z)$ converges for all values of z except at $z=\infty$

TWO SIDED SEQUENCE.

A signal that has finite duration on both the left and right hand sides is known as two sided sequence.

For such type of sequence the ROC is entire z-plane except at $z=0$ and $z=\infty$

Eg: 3

Find the z-transform of the sequence

$$x(n) = \{2, -1, 3, 2, \underset{\uparrow}{1}, 0, 2, 3, -1\}$$

Solution:

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$= 2z^{4} - z^{3} + 3z^{2} + 2z + 1 + 2z^{-2} + 3z^{-3} - z^{-4}.$$

The $X(z)$ converges for all values of z except $z=0$ & $z=\infty$.

Z-TRANSFORM AND ROC OF INFINITE DURATION SEQUENCE.

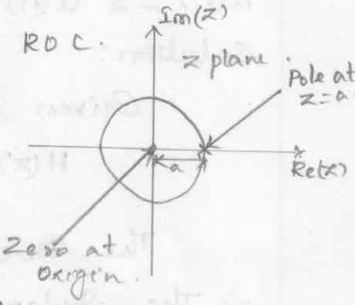
Eg: 4

Determine the z-transform and ROC of the signal $x(n) = a^{n} u(n)$.

Solution

The given signal is causal and of finite duration.

The z-transform of $x(n)$ is given by



$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$= \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} (a z^{-1})^n$$

W.K.T

$$\sum_{n=0}^{\infty} r^n = 1 + r + r^2 + r^3 + \dots = \frac{1}{1-r} \quad \text{if } |r| < 1$$

$$\therefore X(z) = \frac{1}{1 - a z^{-1}} = \frac{z}{z-a} \quad \text{ROC is } |z| > a.$$

i.e) The ROC is the exterior of a circle having radius $|a|$

Eg: 5

Find the z-transform and the ROC of the signal

$$x(n) = -b^n u(-n-1)$$

Solution

The given signal is of infinite duration and anticausal.

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} \qquad u(-n-1) = 0 \text{ for } n \geq 0$$
$$= 1 \text{ for } n \leq 1$$

$$= -\sum_{n=-\infty}^{-1} b^n z^{-n} = -\sum_{n=1}^{\infty} (b^{-n} z^{n})$$

$$= -\left[\sum_{n=0}^{\infty} (b^{-1} z)^{-n} - 1\right] = -\sum_{n=1} (b^{-1} z)^n.$$

The above series converges for $|b^{-1} z| < 1$, i.e) for $|z| < b$.

$$X(z) = -\left[\frac{b^{-1} z}{1 - b^{-1} z}\right]$$

$$= \frac{z}{z-b} \quad \text{ROC: } |z| < b.$$

The ROC is now the interior of a circle.

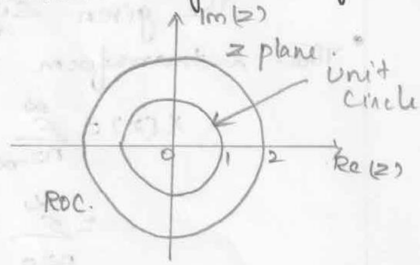
Eg: 6

Find the stability of the System whose impulse response $h(n) = 2^n u(n)$

Solution:

Given $h(n) = 2^n u(n)$

$$H(z) = \frac{z}{z-2} \quad |z| > 2$$



The ROC is $|z| > 2$. It does not contain unit circle.

∴ The system is unstable.

PROPERTIES OF ROC

1. The ROC is a ring or disc in the z-plane centred at the origin.

2. The ROC can not contain any poles

3. If $x(n)$ is a causal sequence the ROC is the entire z-plane except at $z=0$

4. If $x(n)$ is a anticausal sequence the ROC is the entire z-plane except at $z=\infty$

5. If $x(n)$ is a finite duration, two sided sequence the ROC is entire z-plane except at $z=0$ & $z=\infty$.

6. If $x(n)$ is an infinite duration, two sided sequence the ROC will ~~contain~~ consist of a ring in the z-plane, bounded on the interior and exterior by pole, not containing any poles.

7. The ROC of a LTI stable System contains the unit circle.

8. The ROC must be a connected region.

1) Find z-transform of the signal $x(n) = [3(3)^n - 4(2)^n]\, u(n)$.

Solution:

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} [3(3)^n - 4(2)^n]\, u(n) z^{-n}$$

$$= \sum_{n=0}^{\infty} [3(3)^n - 4(2)^n]\, z^{-n}$$

$$= 3 \sum_{n=0}^{\infty} (3z^{-1})^n - 4 \sum_{n=0}^{\infty} (2z^{-1})^n$$

$$= \frac{3}{1-3z^{-1}} - \frac{4}{1-2z^{-1}} \qquad \text{ROC}: |z| > 3.$$

$$= \frac{3z}{z-3} - \frac{4z}{z-2}.$$

2) Find the z-transform of the sequence $x(n) = \left(\frac{1}{3}\right)^{n-1} u(n-1)$

Solution:

$$Z\left[\left(\tfrac{1}{3}\right)^n u(n)\right] = \frac{z}{z-\frac{1}{3}}.$$

Using time shifting property

$$Z[x(n-1)] = z^{-1} X(z).$$

$||^{ly}$

$$Z\left[\left(\tfrac{1}{3}\right)^{n-1} u(n-1)\right] = z^{-1} \frac{z}{z-\frac{1}{3}} = \frac{1}{z-\frac{1}{3}}$$

3) Find the z-transform of the sequence $x(n) = n a^n u(n)$

Solution:

$$Z[a^n u(n)] = \frac{1}{1-az^{-1}} \qquad |z| > |a|$$

Using differentiation property

$$Z[n x(n)] = -z \frac{d}{dz} X(z)$$

$$Z[n a^n u(n)] = -z \frac{d}{dz} \left[\frac{1}{1-az^{-1}}\right]$$

$$= \frac{a z^{-1}}{(1-az^{-1})^2} \quad ; \quad |z| > |a|$$

$$\because -z\frac{d}{dz}\left(\frac{1}{1-az^{-1}}\right) = -z\left[\frac{1}{(1-az^{-1})^2} + a\frac{1}{z^2}\right]$$

$$= \frac{a z^{-1}}{(1-az^{-1})^2}.$$

4) Find inverse z-transform of $x(z) = \log(1-0.5z^{-1})$; $|z| > 0.5$ using Differentiation property.

Solution:

Given $x(z) = \log(1-0.5z^{-1})$

Differentiate on both sides we get

$$\frac{d}{dz}x(z) = \frac{0.5z^{-2}}{1-0.5z^{-1}} \qquad \frac{d}{dx}\log x = \frac{1}{x}.$$

Multiply both sides by $-z$ we obtain

$$-z\frac{d}{dz}x(z) = \frac{-0.5z^{-1}}{1-0.5z^{-1}}$$

$$Z[n\,x(n)] = -z\frac{d}{dz}x(z) = \frac{-0.5z^{-1}}{1-0.5z^{-1}}$$

$$= -0.5\,Z\left[(0.5)^{n-1}u(n-1)\right].$$

$$\left[\because Z[(0.5)^n u(n)] = \frac{1}{1-0.5z^{-1}} \quad Z[(0.5)^{n-1}u(n-1)] = \frac{z^{-1}}{1-0.5z^{-1}}\right]$$

$$\therefore n\,x(n) = -0.5(0.5)^{n-1}u(n-1)$$

$$x(n) = -\frac{(0.5)^n}{n}u(n-1)$$

5) Find the System function and impulse response of the system described by the difference equation $y(n) = \frac{1}{5} y(n-1) + x(n)$

Solution:

Given $y(n) = \frac{1}{5} y(n-1) + x(n)$

Taking z-transform on both sides we get

$$Y(z) = \frac{1}{5} z^{-1} Y(z) + X(z)$$

$$Y(z) - \frac{1}{5} z^{-1} Y(z) = X(z)$$

Now the System function is $H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1 - \frac{1}{5} z^{-1}}$

By taking inverse z-transform

$$h(n) = \left(\frac{1}{5}\right)^n u(n).$$

6) Find the System function and impulse response of the system described by the difference equation

$$y(n) = x(n) + 2x(n-1) - 4x(n-2) + x(n-3)$$

Solution

Given $y(n) = x(n) + 2x(n-1) - 4x(n-2) + x(n-3)$

Taking z-transform on both sides

$$Y(z) = X(z) + 2z^{-1} X(z) - 4z^{-2} X(z) + z^{-3} X(z)$$

$$Y(z) = (1 + 2z^{-1} - 4z^{-2} + z^{-3}) X(z)$$

System function, $H(z) = \frac{Y(z)}{X(z)} = 1 + 2z^{-1} - 4z^{-2} + z^{-3}$

$$= h(0) z^{0} + h(1) z^{-1} + h(2) z^{-2} + h(3) z^{-3}.$$

$h(0) = 1$, $h(1) = 2$, $h(2) = -4$, $h(3) = 1$.

$\therefore$ Impulse response, $h(n) = \{\underset{\uparrow}{1}, 2, -4, 1\}$

5) Find the System function and impulse response of the system described by the difference equation $y(n) = \frac{1}{5}y(n-1) + x(n)$

Solution:

Given $y(n) = \frac{1}{5}y(n-1) + x(n)$

Taking Z-transform on both sides we get

$$Y(Z) = \frac{1}{5}Z^{-1}Y(Z) + X(Z)$$

$$Y(Z) - \frac{1}{5}Z^{-1}Y(Z) = X(Z)$$

Now the System function is $H(Z) = \frac{Y(Z)}{X(Z)} = \frac{1}{1-\frac{1}{5}Z^{-1}}$

By taking inverse Z-transform

$$h(n) = \left(\frac{1}{5}\right)^n u(n).$$

6) Find the System function and impulse response of the system described by the difference equation

$$y(n) = x(n) + 2x(n-1) - 4x(n-2) + x(n-3)$$

Solution

Given $y(n) = x(n) + 2x(n-1) - 4x(n-2) + x(n-3)$

Taking Z-transform on both sides

$$Y(Z) = X(Z) + 2Z^{-1}X(Z) - 4Z^{-2}X(Z) + Z^{-3}X(Z)$$

$$Y(Z) = (1 + 2Z^{-1} - 4Z^{-2} + Z^{-3})\,X(Z)$$

System function, $H(Z) = \frac{Y(Z)}{X(Z)} = 1 + 2Z^{-1} - 4Z^{-2} + Z^{-3}$

$$= h(0)Z^{0} + h(1)Z^{-1} + h(2)Z^{-2} + h(3)Z^{-3}.$$

$h(0) = 1$, $h(1) = 2$, $h(2) = -4$, $h(3) = 1$.

$\therefore$ Impulse response, $h(n) = \{\underset{\uparrow}{1}, 2, -4, 1\}$

$$\frac{z}{12}+\frac{7z^2}{144}+\frac{37}{1728}z^3$$

$12-7z+z^2 \,)\, z$

$(-)\,z \;(+)\,\frac{7z^2}{12} \;(-)\,\frac{z^3}{12}$

$\frac{7z^2}{12}-\frac{z^3}{12}$

$(-)\,\frac{7z^2}{12} \;(+)\,\frac{49z^3}{144} \;(-)\,\frac{7z^4}{144}$

$\frac{37z^3}{144}-\frac{7z^4}{144}$

$\frac{37z^3}{144}-\frac{259z^4}{1728}+\frac{37}{1728}z^5$

$\frac{}{144}$

$$x(z)=\frac{1}{12}z+\frac{7}{144}z^2+\frac{37}{1728}z^3+\cdots$$

$$=\sum_{n=-\infty}^{-1}x(n)z^{-n}$$

$\therefore\; x(0)=0,\quad x(-1)=\frac{1}{12},\quad x(-2)=\frac{7}{144},\quad x(-3)=\frac{37}{1728}$

8) a) Find the inverse z-transform of $x(z)=\dfrac{1+3z^{-1}}{1+3z^{-1}+2z^{-2}}$ $\quad |z|>2$

b) Find the inverse z-transform of $x(z)=\dfrac{z(z^2-4z+5)}{(z-3)(z-1)(z-2)}$ for ROC i) $2<|z|<3$ ii) $|z|>3$ iii) $|z|<1$.

Solution: PARTIAL FRACTION METHOD

a) Given $x(z)=\dfrac{1+3z^{-1}}{1+3z^{-1}+2z^{-2}}$

Multiply numerator and Denominator by z^2

$$x(z)=\frac{z^2+3z}{z^2+3z+2}=\frac{z(z+3)}{(z+1)(z+2)}$$

$$\frac{x(z)}{z}=\frac{z+3}{(z+1)(z+2)}$$

The above equation can be written in partial fraction form as

$$\frac{x(z)}{z}=\frac{C_1}{z+1}+\frac{C_2}{z+2}.$$

$$\frac{x(z)}{z}=\frac{C_1(z+2)+C_2(z+1)}{(z+1)(z+2)}$$

$z+3 \;=\; C_1(z+2)+C_2(z+1) \rightarrow (5)$

To find C_1

Sub $z = -1$, in eqn (1)

$C_1 = 2$.

To find C_2

Sub $z = -2$ in eqn (2)

$C_2 = -1$.

$$\therefore \frac{X(z)}{z} = \frac{2}{z+1} - \frac{1}{z+2}$$

$$X(z) = 2\frac{z}{z+1} - \frac{z}{z+2}.$$

As ROC is $|z| > 2$, the sequence is causal and by inspection, we

$$x(n) = 2(-1)^n u(n) - (-2)^n u(n).$$

b) Given $X(z) = \frac{z(z^2 - 4z + 5)}{(z-1)(z-2)(z-3)}$.

$$\frac{X(z)}{z} = \frac{z^2 - 4z + 5}{(z-1)(z-2)(z-3)}$$

$$= \frac{C_1}{z+1} + \frac{C_2}{z-2} + \frac{C_3}{z-3}.$$

$$z^2 - 4z + 5 = C_1(z-2)(z-3) + C_2(z+1)(z-3) + C_3(z+1)(z-2) \quad \to (2)$$

To find C_1

Sub $z = 1$ in eqn (2).

$C_1 = 1$.

To find C_2

Sub $z = 2$ in eqn (2)

$C_2 = -1$

To find C_3

Sub $z = 3$ in eqn (3)

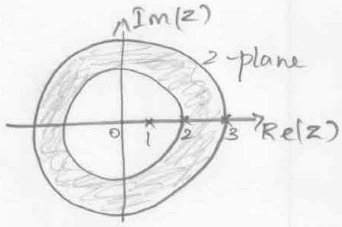
$C_3 = 1$.

$$\therefore \frac{X(z)}{z} = \frac{1}{z-1} + \frac{-1}{z-2} + \frac{1}{z-3}$$

$$X(z) = \frac{z}{} - \frac{1}{} + \frac{1}{}$$

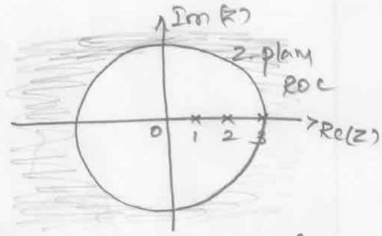
i) In case when the ROC is $2<|z|<3$, the signal $x(n)$ is two sided. The poles $z=1$ and $z=2$ provide the causal part and the pole $z=3$ provides anticausal part.

$$\therefore x(n) = u(n) - (2)^n u(n) - (3)^n u(-n-1)$$



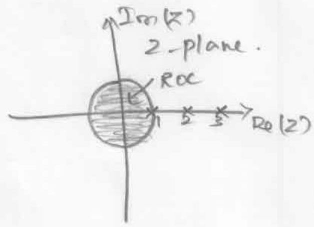
ii) In case when ROC is $|z|>3$, the signal $x(n)$ is causal and all three terms are causal terms.

$$\therefore x(n) = u(n) - (2)^n u(n) + (3)^n u(n)$$



iii) In case when ROC is $|z|<1$, the signal $x(n)$ is anticausal and all the terms are anticausal terms.

$$\therefore x(n) = -u(-n-1) + (2)^n u(-n-1) - (3)^n u(-n-1).$$



CONVOLUTION

i) LINEAR CONVOLUTION

If the input to the system is a unit impulse i.e. $x(n) = \delta(n)$, then impulse response is

$$h(n) = T[\delta(n)]$$

For a linear time-invariant system, if the input sequence $x(n)$ and impulse response $h(n)$ are given, we can find the output $y(n)$ by using the equation

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k)$$

which is known as convolution sum and can be represented as

$$y(n) = x(n) * h(n)$$

PROPERTIES OF CONVOLUTION

i) Commutative Law

$$x(n) * h(n) = h(n) * x(n)$$

ii) Associative Law

$$[x(n) * h_1(n)] * h_2(n) = x(n) * [h_1(n) * h_2(n)]$$

iii) Distributive Law

$$x(n) * [h_1(n) + h_2(n)] = x(n) * h_1(n) + x(n) * h_2(n).$$

ii) CIRCULAR CONVOLUTION

Circular Convolution is denoted by

$$y(n) = x_1(n) \ \text{Ⓝ}\ x_2(n)$$

If $x_1(n)$ contains L samples & $x_2(n)$ contains M samples, then the no. of samples in $y(n)$ is the Max (M, L)

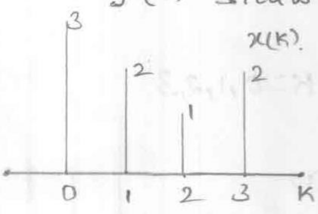
1) Determine the convolution sum of two sequences

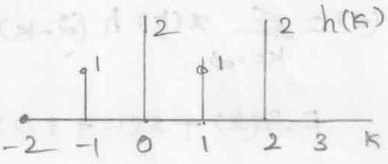
$x(n) = \{3, 2, 1, 2\}$; $h(n) = \{1, \underset{\uparrow}{2}, 1, 2\}$

Solution:

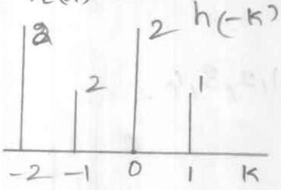
METHOD 1

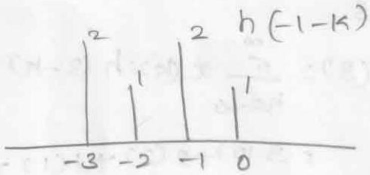
In this example $x(n)$ starts at $n=0$, & $h(n)$ starts at $n=-1$. $y(n)$ starts at $n = n_1 + n_2 = 0 + (-1) = -1$.





Folding $h(k)$ about $k=0$





As starting time to evaluate $y(n)$ is -1.

By convolution formula

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k)$$

$n = -1$.

$$y(-1) = \sum_{k=-\infty}^{\infty} x(k)\, h(-1-k) \qquad k = -3, -2, -1, 0, 1, 2, \ldots$$

$$= 0(2) + 0(1) + 0(2) + 3 + 2(0) + 1(0) + 2(0)$$

$$y(-1) = 3.$$

$n = 0$.

$$y(0) = \sum_{k=-\infty}^{\infty} x(k)\, h(-k) \qquad k = -2, -1, 0, 1, 2, 3.$$

$$= 0(3) + 0(2) + 3(2) + 2(1) + 1(0) + 2(0)$$

$$= 6 + 2$$

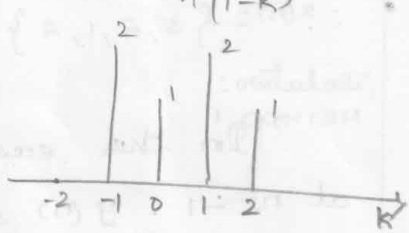
$$y(0) = 8.$$

$n=1$.

$$y(1)=\sum_{k=-\infty}^{\infty} x(k)\,h(1-k) \qquad k=-1,0,1,2,3.$$

$$=0(2)+3(1)+2(2)+1(1)$$

$$=8.$$

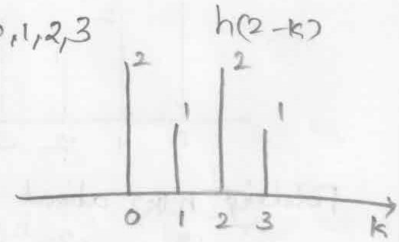


$n=2$

$$y(2)=\sum_{k=-\infty}^{\infty} x(k)\,h(2-k) \qquad k=0,1,2,3$$

$$=3(2)+2(1)+1(2)+2(1)$$

$$=12$$

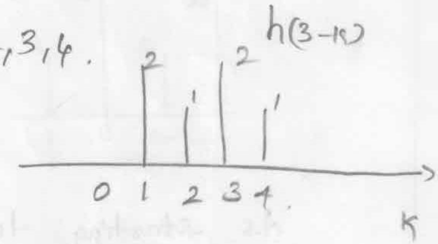


$n=3$

$$y(3)=\sum_{k=-\infty}^{\infty} x(k)\,h(3-k) \qquad k=0,1,2,3,4.$$

$$=3(0)+2(2)+1(1)+2(2)$$

$$=9.$$

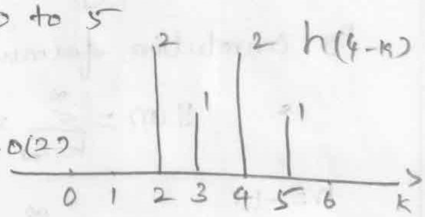


$n=4$.

$$y(4)=\sum_{k=-\infty}^{\infty} x(k)\,h(4-k) \qquad k=0 \text{ to } 5$$

$$=3(0)+2(0)+1(2)+2(1)+0(2)$$

$$=4$$

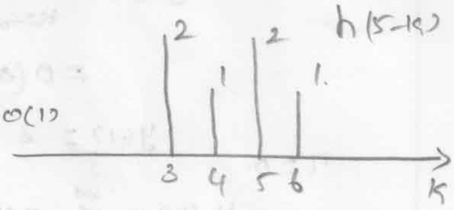


$n=5$.

$$y(5)=\sum_{k=-\infty}^{\infty} x(k)\,h(5-k)$$

$$=3(0)+2(0)+1(0)+2(2)+0(1)$$
$$+0(2)+0(1)$$

$$=4.$$



$\therefore y(n)=\{3,\underset{\uparrow}{8},8,12,9,4,4\}$ $\quad\therefore$ Length of the o/p Sequence $L+M-1$.

METHOD 2

	x(n) → 3	2	1	2
h(n): 1	3	2	1	2
2	6	4	2	4
1	3	2	1	2
2	6	4	2	4.

$y(n) = \{3, \underset{\uparrow}{8}, 8, 12, 9, 4, 4\}$

$n=0.$

2) Find the convolution of the signals

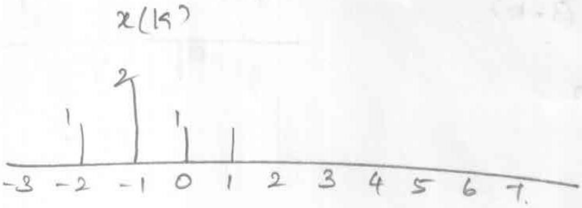
$x(n) = 1 \quad n = -2, 0, 1$

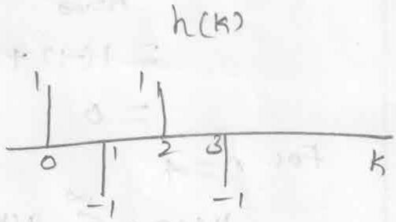
$\quad = 2 \quad n = -1$

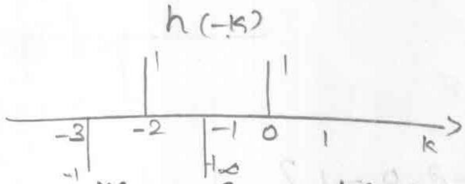
$\quad = 0 \quad$ elsewhere

$h(n) = \delta(n) - \delta(n-1) + \delta(n-2) - \delta(n-3)$

Solution:

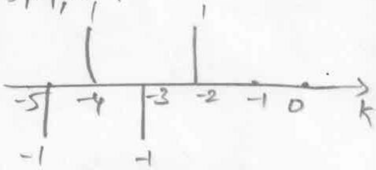






$n = -2. \quad y(k) = \sum_{k=-\infty}^{\infty} x(k) h(n-k)$

$K = -5, -4, \ldots 1$



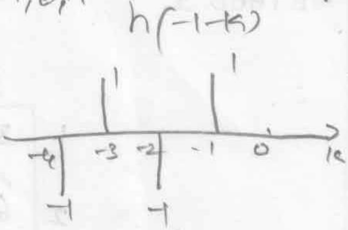
$y(-2) = \sum_{k=-\infty}^{\infty} x(k) h(-2-k)$

$= 1(1)$

For $n = -1$ $\quad k = -4, -3, -2, -1, 0, 1$

$$y(-1) = \sum_{k=-\infty}^{\infty} x(k)\, h(-1-k)$$

$$= 1(-1) + 2(1) = 1$$



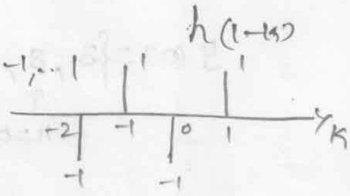
For $n = 0$

$$y(0) = \sum_{k=-\infty}^{\infty} x(k)\, h(-k) \qquad k = -3, -2, -1, \dots +3.$$

$$= 1(1) + 2(-1) + 1(1) = 0$$

For $n = 1$

$$y(1) = \sum_{k=-\infty}^{\infty} x(k)\, h(1-k) \qquad k = -2, -1, \dots 1$$

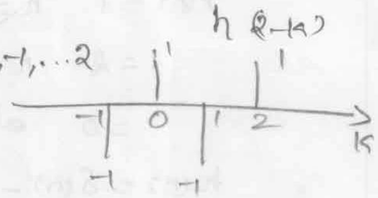


$$= 1(-1) + 2(1) + 1(-1) + 1(1)$$

$$= 1$$

For $n = 2$

$$y(2) = \sum_{k=-\infty}^{\infty} x(k)\, h(2-k) \qquad k = -2, -1, \dots 2$$

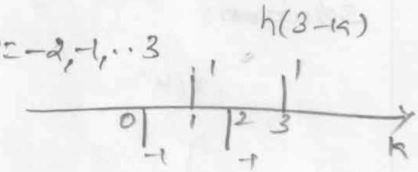


$$= 2(-1) + 1(1) + 1(-1)$$

$$= -2$$

For $n = 3$ $\quad k = -2, -1, \dots 3$

$$y(3) = \sum_{k=-\infty}^{\infty} x(k)\, h(3-k)$$

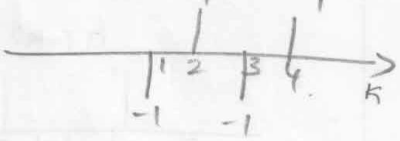


$$= 1(-1) + 1(1)$$

$$= 0$$

For $n = 4$ $\quad k = -2, -1, 0, \dots 4.$

$$y(4) = \sum_{k=-\infty}^{\infty} x(k)\, h(4-k)$$



$$= 1(-1) = -1$$

$$y(n) = \{1, 1, \underset{\uparrow}{0}, 1, -2, 0, -1\}$$

$n = 0$

3) Find the convolution of two finite sequences

$h(n) = a^n u(n)$ for all n i) when $a \neq b$

$x(n) = b^n u(n)$ for all n ii) when $a = b$

Solution

The impulse response $h(n) = 0$ for $n<0$, so the given system is causal and $x(n)=0$ for $n<0$, hence the sequence is a causal sequence.

$$y(n) = \sum_{k=0}^{n} x(k) h(n-k)$$

$$= \sum_{k=0}^{n} b^k a^{n-k} = a^n \sum_{k=0}^{n} \left(\frac{b}{a}\right)^k = a^n \left[1 + \frac{b}{a} + \frac{b^2}{a^2} + \cdots (n+1) \text{ terms}\right]$$

$$y(n) = a^n \left[\frac{1-(b/a)^{n+1}}{1-b/a}\right] \qquad \because \sum_{n=0}^{N-1} a^n = \frac{1-a^N}{1-a}$$

If $a = b$,

$$y(n) = a^n \sum_{k=0}^{n} (1)^k = a^n (n+1) \qquad \because 1+1+1+\cdots (n+1) \text{ terms} = n+1$$

4) Find $y(n)$ if $x(n) = n+2$ for $0 \leq n \leq 3$

$h(n) = a^n u(n)$ for all n

Solution:

$$y(n) = \sum_{k=-\infty}^{\infty} x(k) h(n-k)$$

Given $x(n) = n+2$ for $0 \leq n \leq 3$

$h(n) = a^n u(n)$ for all n

$h(n) = 0$ for $n<0$. so the system is causal

$x(n)$ is a causal finite sequence whose value is zero for $n>3$.

$$\therefore y(n) = \sum_{k=0}^{3} x(k) h(n-k)$$

$$= \sum_{k=0}^{3} (k+2)\, a^{n-k} u(n-k)$$

$$= 2a^n u(n) + 3a^{n-1} u(n-1) + 4a^{n-2} u(n-2) + 5a^{n-3} u(n-3)$$

5) Determine the response of the relaxed system characterized by the impulse response $h(n) = (1/2)^n u(n)$ to the input signal $x(n) = 2^n u(n)$.

Solution:

Given $x(n) = 2^n u(n)$; $h(n) = (1/2)^n u(n)$

A causal signal is applied to a causal system.

$$\therefore y(n) = \sum_{k=0}^{n} x(k)\, h(n-k)$$

$$= \sum_{k=0}^{n} 2^k \left(\frac{1}{2}\right)^{n-k}$$

$$= \left(\frac{1}{2}\right)^n \sum_{k=0}^{n} 2^k \left(\frac{1}{2}\right)^{-k} = \left(\frac{1}{2}\right)^n \sum_{k=0}^{n} 2^{2k}$$

$$= (1/2)^n \left[1 + 2^2 + 2^4 + 2^6 - - - (n+1) \text{ terms} \right].$$

$$= (1/2)^n \left[\frac{(2^2)^{n+1} - 1}{2^2 - 1} \right]$$

$$= \left(\frac{1}{2}\right)^n \left[\frac{4 \cdot 4^n - 1}{4 - 1} \right]$$

$$y(n) = \left(\frac{1}{2}\right)^n \left(\frac{4^{n+1} - 1}{3} \right)$$

6) Find the circular convolution of two finite duration sequence

$x_1(n) = \{1, -1, -2, 3, -1\}$; $x_2(n) = \{1, 2, 3\}$.

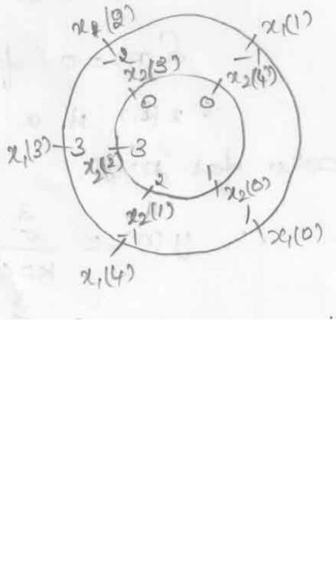
Solution:

METHOD I

Concentric Circle Method.

$x_1(n) = \{1, -1, -2, 3, -1\}$,

$x_2(n) = \{1, 2, 3, 0, 0\}$

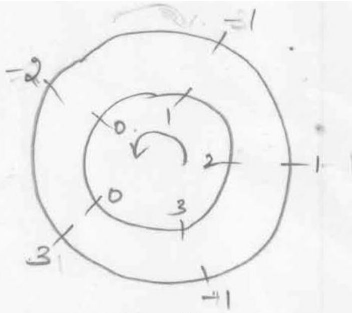


$y(0) = 1(1) + 0(-1) + 0(-2) + 3(3) + 2(-1)$

$y(0) = 8$

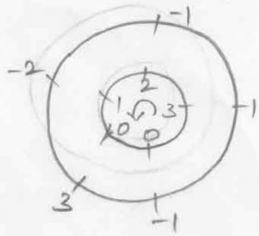
$y(1) = 1(2) + (-1)(1) + (-2)(0) + 3(0) + 3(-1)$

$y(1) = -2$



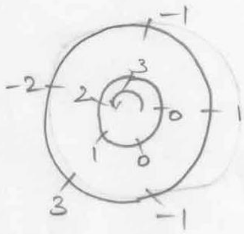
$y(2) = 3(1) + 2(-1) + 1(-2) + 0(3) + (0)(-1)$

$y(2) = -1$



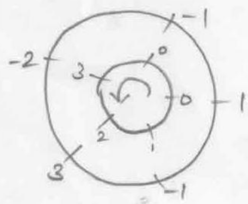
$y(3) = 0(1) + 3(-1) + 2(-2) + 1(3) + (-1)(0)$

$= -4.$



$y(4) = 0(1) + 0(-1) + 3(-2) + 3(2) + 1(-1)$

$= -1$



METHOD 2

Matrix Method

$x_1(n) = \{1, -1, -2, 3, -1\}$ $\quad x_2(n) = \{1, 2, 3, 0, 0\}$

$x_2(n)$

$$\begin{bmatrix} 1 & 0 & 0 & 3 & 2 \\ 2 & 1 & 0 & 0 & 3 \\ 3 & 2 & 1 & 0 & 0 \\ 0 & 3 & 2 & 1 & 0 \\ 0 & 0 & 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ -2 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 8 \\ -2 \\ -1 \\ -4 \\ -1 \end{bmatrix}$$

$\therefore y(n) = \{8, -2, -1, -4, 1\}$

$$\begin{bmatrix} x_2(0) & x_2(4) & x_2(3) & x_2(2) & x_2(1) \\ x_2(1) & x_2(0) & x_2(4) & x_2(3) & x_2(2) \\ x_2(2) & x_2(1) & x_2(0) & x_2(4) & x_2(3) \\ x_2(3) & x_2(2) & x_2(1) & x_0(0) & x_2(4) \\ x_2(4) & x_2(3) & x_2(2) & x_2(1) & x_2(0) \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_1(1) \\ x_1(2) \\ x_1(3) \\ x_1(4) \end{bmatrix} = \begin{bmatrix} y_1(0 \\ y_1(1 \\ y_1(2 \\ y_1(3 \\ y_1(4 \end{bmatrix}$$

CORRELATION

Correlation is basically used to Compare two Signals

It is a measure of Similarity between two Signals.

Types

i) AUTO CORRELATION

It is a measure of Similarity among the same signal.

The autocorrelation of a Sequence $x(n)$ is defined by

$$\gamma_{xx}(l) = \sum_{n=-\infty}^{\infty} x(n)\, x(n-l)$$

(or)

$$\gamma_{xx}(l) = \sum_{n=-\infty}^{\infty} x(n+l)\, x(n)$$

If the time shift $l=0$, then we have

$$\gamma_{xx}(0) = \sum_{n=-\infty}^{\infty} x^2(n)$$

ii) CROSS CORRELATION

It is a measure of Similarity between two different signals.

The cross correlation between the signals $x(n)$ & $y(n)$ is given by

$$\gamma_{xy}(l) = \sum_{n=-\infty}^{\infty} x(n)\, y(n-l) \rightarrow \text{ⓐ} \quad l = 0, \pm1, \pm2, \ldots$$

(or)

$$\gamma_{yx}(l) = \sum_{n=-\infty}^{\infty} y(n)\, x(n-l)$$

$$= \sum_{n=-\infty}^{\infty} y(n+l)\, x(n) \rightarrow \text{ⓑ}$$

If the time shift $l=0$, then we get

$$\gamma_{xy}(0) = \gamma_{yx}(0) = \sum_{n=-\infty}^{\infty} x(n)\,y(n)$$

Comparing equations ⓐ & ⓑ we find that

$$\gamma_{xy}(l) = \gamma_{yx}(-l)$$

Where $\gamma_{yx}(-l)$ is the folded version of $\gamma_{yx}(l)$ about $l=0$.

We can rewrite the equation ⓐ

$$\gamma_{xy}(l) = \sum_{n=-\infty}^{\infty} x(n)\,y[-(l-n)]$$

$$= x(l) * y(-l)$$

From the above equation, we find that the correlation process is essentially the convolution of two data sequences in which one of the sequences has been reversed.

1) Find the cross correlation of two finite length sequences $x(n) = \{1,2,1,1\}$ $y(n) = \{1,1,2,1\}$.

Solution:

$\gamma_{xy}(l) = x(l) * y(-l)$ $\quad x(l) = \{1,2,1,1\}$ $\quad y(-l) = \{1,2,1,\underset{\uparrow}{1}\}$

$x(l)$ \ $y(-l)$	1	2	1	1
1	1	2	1	1
2	2	4	2	2
1	1	2	1	1
1	1	2	1	1

$\gamma_{xy}(l) = \{1, 4, 6, \underset{\uparrow}{6}, 5, 2, 1\}$

UNIT-3

TWIDDLE FACTOR (OR) WINDOW FUNCTION

The twiddle factor or window function is given by

$$W_N = e^{-2\pi/N}$$

Hence the DFT Pair is

$$X(K) = \sum_{n=0}^{N-1} x(n) W_N^{nK}, \quad k=0,1,\dots N-1$$

$$x(n) = \frac{1}{N}\sum_{K=0}^{N-1} X(K) W_N^{-nk}, \quad n=0,1,\dots N-1$$

RELATIONSHIP OF THE DFT TO OTHER TRANSFORM

i) RELATION TO THE FOURIER TRANSFORM

The FT $X(e^{j\omega})$ of a finite duration sequence $x(n)$ having length N is given by

$$X(e^{j\omega}) = \sum_{n=0}^{N-1} x(n) e^{-j\omega n} \longrightarrow (3)$$

where $X(e^{j\omega})$ is a continuous function of ω.

The DFT of $x(n)$ is given by

$$X(K) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N}, \quad K=0,1,2,\dots N-1 \longrightarrow (4)$$

Comparing eqn (3) & (4), DFT of $x(n)$ is a sampled version of the Fourier transform of the sequence and is given by

$$X(K) = X(e^{j\omega})\Big|_{\omega=\frac{2\pi k}{N}} \quad K=0,1,2,\dots N-1$$

ii) RELATIONSHIP TO THE Z-TRANSFORM

Let us consider a sequence $x(n)$ of finite duration N with Z-transform

$$X(Z) = \sum_{n=0}^{N-1} x(n) Z^{-n} \longrightarrow (5)$$

we have

$$x(n) = \frac{1}{N}\sum_{K=0}^{N-1} X(K) e^{j2\pi kn/N} \longrightarrow (6)$$

FREQUENCY TRANSFORMATION

INTRODUCTION TO DFT

The Discrete Fourier Transform (DFT) is a powerful computation tool which allows us to evaluate the Fourier Transform $X(e^{j\omega})$ on a digital computer or specially designed hardware.

DFT is defined only for sequence of finite length.

DFT PAIR

The DFT of a discrete signal $x(n)$ is given by

$$X(e^{j\omega}) X(K) = \sum_{n=0}^{N-1} x(n) e^{-j\omega n}$$

If we sample $X(e^{j\omega})$ at N equally spaced points over $0 \le \omega \le 2\pi$, we obtain

$$X(K) = X(e^{j\omega})\Big|_{\omega = 2\pi k/N} = \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N}, \quad 0 \le K \le N-1 \longrightarrow (1)$$

Which is called an N point DFT.

Where, $X(K)$ – Signal in Frequency domain

$x(n)$ – Signal in time domain

N – No. of samples (or) Length of the sequence.

The IDFT of $X(K)$ is given by

$$x(n) = \frac{1}{N} \sum_{K=0}^{N-1} X(K) e^{\frac{j2\pi kn}{N}}, \quad K = 0,1,2,\ldots (N-1) \longrightarrow (2)$$

Equations (1) & (2) are together called the DFT Pair.

Substituting Eqn (6) in (5) we get

$$X(Z) = \sum_{n=0}^{N-1} \left[\frac{1}{N} \sum_{K=0}^{N-1} X(K) e^{j2\pi kn/N} \right] Z^{-n}$$

$$= \frac{1}{N} \sum_{K=0}^{N-1} X(K) \sum_{n=0}^{N-1} \left(e^{j2\pi k/N} Z^{-1} \right)^{n}$$

$$= \frac{1 - Z^{-N}}{N} \sum_{K=0}^{N-1} \frac{X(K)}{1 - e^{j2\pi k/N} Z^{-1}}$$

PROPERTIES OF THE DISCRETE FOURIER TRANSFORM

i) PERIODICITY

If $X(K)$ is N-point DFT of a finite duration sequence $x(n)$ then

$$x(n+N) = x(n) \quad \text{for all } n$$

$$X(K+N) = X(K) \quad \text{for all } k$$

ii) LINEARITY

$$DFT\left[a\, x_1(n) + b\, x_2(n)\right] = a\, X_1(K) + b\, X_2(K).$$

$$x_1(n) \overset{DFT}{\longleftrightarrow} X_1(K)$$

$$x_2(n) \overset{DFT}{\longleftrightarrow} X_2(K)$$

iii) THE REVERSAL OF THE SEQUENCE

The time reversal of an N-point sequence $x(n)$ is attained by wrapping the sequence $x(n)$ around the circle in clockwise direction. It is denoted as $x((-n))_N$ and

$$x((-n))_N = x(N-n) \qquad 0 \le n \le N-1.$$

If $DFT[x(n)] = X(K)$ then

$$DFT\left[x((-n))_N\right] = DFT\left[x(N-n)\right]$$

$$= X((-K))_N = X(N-K)$$

PROOF

$$DFT[x(N-n)] = \sum_{n=0}^{N-1} x(N-n)\, e^{-j2\pi kn/N}$$

changing the index from n to $m = N-n$ we get

$$DFT[x(N-m)] = \sum_{m=0}^{N-1} x(m)\, e^{-j2\pi k(N-m)/N}$$

$$= \sum_{m=0}^{N-1} x(m)\, e^{j2\pi km/N} \quad \because e^{j2\pi K} = 1 \text{ for } K = 0, 1, 2 \ldots$$

$$= \sum_{m=0}^{N-1} x(m)\, e^{-j2\pi m(N-K)/N}$$

$$= X(N-K)$$

iv) CIRCULAR FREQUENCY SHIFT

If $DFT[x(n)] = X(k)$ then

$$DFT\left[x(n)\, e^{j2\pi ln/N}\right] = X((k-l))_N$$

PROOF:

$$DFT\left[x(n)\, e^{j2\pi ln/N}\right] = \sum_{n=0}^{N-1} x(n)\, e^{j2\pi ln/N}\, e^{-j2\pi kn/N}$$

$$= \sum_{n=0}^{N-1} x(n)\, e^{-j2\pi n(k-l)/N}$$

$$= \sum_{n=0}^{N-1} x(n)\, e^{-j2\pi n(N+k-l)/N}$$

$$= X(N+k-l)$$

$$= X((k-l))_N$$

$x(n) = [x(0), x(1), \ldots x(N-1)]$

$x((n-1))_N = [x(N-1), x(0), \ldots x(N-2)]$

$x((n-N))_N = [x(0), x(1), \ldots x(N-1)]$

A shift $k = N$ results in original seq $x(n)$

i.e., $x((n-N))_N = x(n)$

$\parallel^{ly}$ $x((n-m))_N = x(N-M+n)$

v) COMPLEX CONJUGATE PROPERTY

If $DFT[x(n)] = X(k)$

then,

$$DFT[x^*(n)] = X^*(N-k) = X^*((-k))_N.$$

PROOF

$$DFT[x^*(n)] = \sum_{n=0}^{N-1} x^*(n)\, e^{-j2\pi kn/N}$$

$$= \left[\sum_{n=0}^{N-1} x(n)\, e^{j2\pi kn/N}\right]^*$$

$$= \left[\sum_{n=0}^{N-1} x(n)\, e^{-j2\pi n(N-k)/N}\right]^*$$

$$DFT[x^*(n)] = X^*(N-k)$$

$$DFT[x^*(N-n)] = X^*(K)$$

PROOF

$$IDFT[X^*(K)] = \frac{1}{N}\sum_{K=0}^{N-1} X^*(K)\, e^{j2\pi kn/N}$$

$$= \frac{1}{N}\left[\sum_{K=0}^{N-1} X(K)\, e^{-j2\pi kn/N}\right]^*$$

$$= \frac{1}{N}\left[\sum_{K=0}^{N-1} X(K)\, e^{-j2\pi k(N-n)/N}\right]^*$$

$$= x^*(N-n)$$

Therefore,

$$DFT[x^*(N-n)] = X^*(K)$$

CIRCULAR CONVOLUTION

Let $x_1(n)$ and $x_2(n)$ are finite duration sequences both of length N with DFTs $X_1(K)$ and $X_2(K)$. Now we find a sequence $x_3(n)$ for which the DFT is $X_3(K)$. Where

$$X_3(K) = X_1(K)\, X_2(K)$$

$$DFT\left[x_1(n) \textcircled{N} x_2(n)\right] = X_1(K) X_2(K).$$

vii) CIRCULAR CORRELATION

For complex-valued sequences $x(n)$ and $y(n)$ if

$$DFT\left[x(n)\right] = X(K) \quad \text{and}$$

$$DFT\left[y(n)\right] = Y(K).$$

then

$$DFT\left[\tilde{r}_{xy}(l)\right] = DFT\left[\sum_{n=0}^{N-1} x(n) y^*((n-l))_N\right] = X(K) Y^*(K)$$

where $\tilde{r}_{xy}(l)$ is the circular cross correlation sequence

viii) MULTIPLICATION OF TWO SEQUENCES

If $DFT\left[x_1(n)\right] = X_1(K)$

and $DFT\left[x_2(n)\right] = X_2(K)$

then

$$DFT\left[x_1(n) x_2(n)\right] = \frac{1}{N}\left[X_1(K) \textcircled{N} X_2(K)\right]$$

ix) PARSEVAL'S THEOREM

If $DFT\left[x(n)\right] = X(K)$

and $DFT\left[y(n)\right] = Y(K)$

then

$$\sum_{n=0}^{N-1} x(n) y^*(n) = \frac{1}{N} \sum_{K=0}^{N-1} X(K) Y^*(K).$$

(1) Determine the 4-Point DFT and IDFT of the given Signal

$$x(n) = \begin{cases} 1, & 0 \le n \le 3 \\ 0, & \text{elsewhere} \end{cases}$$

Sol!

The i/p sequence is, $x(n) = \{1, 1, 1, 1\}$. For this problem $L = N = 4$.

DFT of $x(n)$ is, $X(K) = \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi}{N}nk}$, $K = 0, 1, \ldots (N-1)$

$$X(K) = \sum_{n=0}^{3} x(n) e^{-j\frac{2\pi}{4}nk}, \quad K = 0, 1, 2, 3$$

$$= \sum_{n=0}^{3} x(n) e^{-j\frac{\pi}{2}nk}, \quad K = 0, 1, 2, 3$$

Let $K = 0$,

$$X(0) = \sum_{n=0}^{3} x(n) e^{-j\pi/2\, n(0)} \quad (=1)$$

$$= x(0) + x(1) + x(2) + x(3)$$

$$X(0) = 1 + 1 + 1 + 1 = 4$$

For $K = 1$

$$X(1) = \sum_{n=0}^{3} x(n) e^{-j\pi/2\, n}$$

$$= x(0) e^{-j\pi/2(0)} + x(1) e^{-j\pi/2} + x(2) e^{-j\pi} + x(3) e^{-j3\pi/2}$$

$$= 1 + 1\left[\cos(\pi/2) - j\sin(\pi/2)\right] + 1\left[\cos(\pi) - j\sin(\pi)\right] + 1\left[\cos(3\pi/2) - j\sin(3\pi/2)\right]$$

$$= 1 + (0 - j) + (-1 - j0) + (0 + j)$$

$$X(1) = 1 - j - 1 + j = 0$$

For $k = 2$

$$X(2) = \sum_{n=0}^{3} x(n) e^{-j\pi n}$$

$$= x(0) + x(1) e^{-j\pi} + x(2) e^{-j2\pi} + x(3) e^{-j3\pi}$$

$= 1 + (\cos\pi - j\sin\pi) + (\cos 2\pi - j\sin 2\pi) + (\cos 3\pi - j\sin 3\pi)$

$= 1 + (0 - j) + (-1 - j0) + (0 + j)$

$= 1 - j - 1 + j$

$= 1 + (-1 - 0) + (1 - 0) + (-1 - 0)$

$X(2) = 1 - 1 + 1 - 1 = 0.$

For $k = 3$

$$X(3) = \sum_{n=0}^{3} x(n) e^{-j\frac{3\pi}{2}n}$$

$$= x(0) + x(1) e^{-j\frac{3\pi}{2}} + x(2) e^{-j3\pi} + x(3) e^{-j\frac{9\pi}{2}}$$

$$= 1 + \left[\cos\left(\frac{3\pi}{2}\right) - j\sin\left(\frac{3\pi}{2}\right)\right] + (\cos 3\pi - j\sin 3\pi) + \left[\cos\left(\frac{9\pi}{2}\right) - j\sin\left(\frac{9\pi}{2}\right)\right]$$

$= 1 + (0 + j) + (-1 - 0) + (0 - j)$

$X(3) = 1 + j - 1 - j = 0.$

$\therefore$ The DFT of $x(n)$ is, $X(K) = \{4, 0, 0, 0\}$

To find IDFT

$$x(n) = \frac{1}{N} \sum_{K=0}^{N-1} X(K) e^{j2\pi/N\, nk}, \quad n = 0, 1, \ldots, (N-1)$$

$$x(n) = \frac{1}{4} \sum_{K=0}^{N-1} X(K) e^{j\pi/2\, nk}, \quad n = 0, 1, 2, 3.$$

For $n = 0$

$$x(0) = \frac{1}{4} \sum_{K=0}^{3} X(K) e^{0}, \quad n = 0, 1, 2, 3$$

$$= \frac{1}{4} \left[X(0) + X(1) + X(2) + X(3) \right]$$

$$= \frac{1}{4} [4 + 0 + 0 + 0]$$

$x(0) = 1$

For $n=1$

$$x(1)=\frac{1}{4}\sum_{K=0}^{3}X(K)e^{j\frac{\pi}{2}k}\quad,\ n=0,1,2,3$$

$$=\frac{1}{4}\left[X(0)e^{0}+X(1)e^{j\frac{\pi}{2}}+X(2)e^{j\pi}+X(3)e^{j\frac{3\pi}{2}}\right]$$

$$=\frac{1}{4}\left[4+0+0+0\right]$$

$$x(1)=1$$

For $n=2$,

$$x(2)=\frac{1}{4}\sum_{K=0}^{3}X(K)e^{j\pi k}\quad,\ n=0,1,2,3$$

$$x(2)=\frac{1}{4}\left[X(0)e^{0}+X(1)e^{j\pi}+X(2)e^{j2\pi}+X(3)e^{j3\pi}\right]$$

$$=\frac{1}{4}\left[4+0+0+0\right]$$

$$x(2)=1$$

For $n=3$

$$x(3)=\frac{1}{4}\left[\sum_{K=0}^{3}X(K)e^{j\frac{3\pi}{2}k}\right],\ n=0,1,2,3$$

$$=\frac{1}{4}\left[X(0)e^{0}+X(1)e^{j\frac{3\pi}{2}}+X(2)e^{j3\pi}+X(3)e^{j\frac{9\pi}{2}}\right]$$

$$=\frac{1}{4}\left[4+0+0+0\right]$$

$$x(3)=1$$

$\therefore$ The IDFT is, $x(n)=\{1,1,1,1\}$

1) Find the DFT of a Sequence $x(n) = \{1, 1, 0, 0\}$ and find the IDFT of $Y(K) = \{1, 0, 1, 0\}$

Solution

Let us assume $N = L = 4$

DFT of $x(n)$ is given as

$$X(K) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi nk/N} \qquad K = 0, 1, \ldots N-1.$$

$K=0$.

$$X(0) = \sum_{n=0}^{3} x(n) = x(0) + x(1) + x(2) + x(3)$$

$$= 1 + 1 + 0 + 0$$

$$= 2$$

$K=1$

$$X(1) = \sum_{n=0}^{3} x(n) e^{-j\pi n/2} = x(0) + x(1) e^{-j\pi/2} + x(2) e^{-j\pi} + x(3) e^{-j3\pi/2}$$

$$= 1 + \cos \pi/2 - j \sin \pi/2$$

$$= 1 - j$$

$K=2$

$$X(2) = \sum_{n=0}^{3} x(n) e^{-j\pi n} = x(0) + x(1) e^{-j\pi} + x(2) e^{-j2\pi} + x(3) e^{-j3\pi}$$

$$= 1 + \cos \pi - j \sin \pi$$

$$= 1 - 1 = 0$$

$K=3$.

$$X(3) = \sum_{n=0}^{3} x(n) e^{-j3\pi n/2} = x(0) + x(1) e^{-j3\pi/2} + x(2) e^{-j3\pi} + x(3) e^{-j3\pi/2}$$

$$= 1 + \cos 3\pi/2 - j \sin 3\pi/2$$

$$= 1 + j$$

$$X(K) = \{2, 1-j, 0, 1+j\}$$

IDFT: $\quad y(n) = \frac{1}{N} \sum_{k=0}^{N-1} Y(K) e^{j2\pi nk/N}, \quad n = 0, 1, \ldots N-1$

$$y(0) = \frac{1}{4} \sum_{k=0}^{3} Y(K)$$

$$= \frac{1}{4} [Y(0) + Y(1) + Y(2) + Y(3)] = 1/4 [1 + 0 + 1 + 0]$$

$$y(1) = \frac{1}{N}\sum_{k=0}^{3} Y(k)\, e^{j\pi k/2}$$

$$= \frac{1}{4}\left[Y(0) + Y(1)e^{j\pi/2} + Y(2)e^{j\pi} + Y(3)e^{j3\pi/2}\right]$$

$$= \frac{1}{4}\left[1 + 0 + \cos\pi + j\sin\pi + 0\right]$$

$$= \frac{1}{4}\left[1 + 0 - 1 + 0\right] = 0.$$

$$y(2) = \frac{1}{4}\left[Y(0) + Y(1)e^{j\pi} + Y(2)e^{j2\pi} + Y(3)e^{j3\pi}\right]$$

$$= \frac{1}{4}\left[1 + 0 + \cos 2\pi + j\sin 2\pi + 0\right]$$

$$= \frac{1}{4}\left[1 + 0 + 1 + 0\right] = 0.5$$

$$y(3) = \frac{1}{4}\left[Y(0) + Y(1)e^{j3\pi/2} + Y(2)e^{j3\pi} + Y(3)e^{j9\pi/2}\right]$$

$$= \frac{1}{4}\left[1 + 0 + \cos 3\pi + j\sin 3\pi + 0\right]$$

$$= \frac{1}{4}\left[1 + 0 + (-1) + 0\right] = 0.$$

$$y(n) = \{0.5, 0, 0.5, 0\}$$

2) Find the DFT of a sequence

$$x(n) = 1 \quad \text{for } 0 \le n \le 2$$
$$= 0 \quad \text{for otherwise.}$$

for i) $N = 4$ ii) $N = 8$

Solution

Given $x(n) = \{1, 1, 1\}$.

Length of this sequence is $L = 3$.

But $N = 4$, $\therefore$ One zero is added to the sequence ie) $N-L$ zeros

$$x(n) = \{1, 1, 1, 0\}.$$

DFT

$$X(k) = \sum_{n=0}^{N-1} x(n)\, e^{-j2\pi kn/N} \qquad k = 0, 1, \ldots, N-1.$$

For $N = 4$

$$X(k) = \sum_{n=0}^{3} x(n)\, e^{-j\pi nk/2} \qquad k = 0, 1, 2, 3$$

$k = 0$

$$X(0) = x(0) + x(1)\, e^{j} + x(2) + x(3)$$

$$= 3$$

$k = 1$

$$X(1) = x(0)\, e^{-0} + x(1)\, e^{-j\pi/2} + x(2)\, e^{-j\pi} + x(3)\, e^{-j3\pi/2}.$$

$$= 1 + \cos \pi/2 - j\sin \pi/2 + \cos\pi - j\sin\pi + 0$$

$$= 1 - j - 1 = -j$$

$k = 2$

$$X(2) = x(0) + x(1)\, e^{-j\pi} + x(2)\, e^{-2\pi} + x(3)\, e^{-j3\pi}.$$

$$= 1 + \cos\pi - j\sin\pi + \cos 2\pi - j\sin 2\pi + 0$$

$$= 1 - 1 + 1 = 1$$

$k = 3.$

$$X(3) = x(0) + x(1)\, e^{-j3\pi/2} + x(2)\, e^{-j3\pi} + x(3)\, e^{-9\pi/2}$$

$$= 1 + \cos 3\pi/2 - j\sin 3\pi/2 + \cos 3\pi - j\sin 3\pi + 0$$

$$= 1 + j - 1 = j$$

$$\therefore X(k) = \{3, -j, 1, j\}.$$

For $N = 8$, Add 5 zeros ies $N-L$ zeros. eo $x(n)$.

$$x(n) = \{1, 1, 1, 0, 0, 0, 0, 0\}$$

$$X(k) = \sum_{n=0}^{7} x(n)\, e^{-j\pi nk/2} \qquad k = 0, 1, 2, \ldots, 7.$$

For $k = 0$

$$X(0) = \sum_{n=0}^{7} x(n)\, e^{0}.$$

$$= x(0) + x(1) + x(2) + x(3) + x(4) + x(5) + x(6) + x(7).$$

$$= 3$$

For $k=1$

$$X(1) = \sum_{n=0}^{7} x(n) e^{-j\pi n/4}$$

$$= x(0) + x(1) e^{-j\pi/4} + x(2) e^{-j\pi/2}.$$

$$= 1 + 0.707 - j0.707 + 0 - j$$

$$= 1.707 - j1.707$$

For $k=2$

$$X(2) = \sum_{n=0}^{7} x(n) e^{-j\pi n/2}.$$

$$= x(0) + x(1) e^{-j\pi/2} + x(2) e^{-j\pi}$$

$$= 1 + \cos \pi/2 - j\sin \pi/2 + \cos \pi - j\sin \pi$$

$$= 1 - j - 1 = -j$$

For $k=3$

$$X(3) = \sum_{n=0}^{3} x(n) e^{-j3\pi n/4}$$

$$= x(0) + x(1) e^{-j3\pi/4} + x(2) e^{-j3\pi/2}$$

$$= 1 + \cos 3\pi/4 - j\sin 3\pi/4 + \cos 3\pi/2 - j\sin 3\pi/2$$

$$= 1 - 0.707 - j0.707 + j$$

$$= 0.293 + j0.293$$

For $k=4$

$$X(4) = \sum_{n=0}^{7} x(n) e^{-j\pi n}$$

$$= x(0) + x(1) e^{-j\pi} + x(2) e^{-j2\pi}$$

$$= 1 + \cos \pi - j\sin \pi + \cos 2\pi - j\sin 2\pi$$

$$= 1 - 1 + 1 = 1$$

For $k=5$

$$X(5) = \sum_{n=0}^{7} x(n) e^{-j5\pi/4}$$

$$= x(0) + x(1) e^{-j5\pi/4} + x(2) e^{-j5\pi/2}$$

$$= 1 + \cos 5\pi/4 - j\sin 5\pi/4 + \cos 5\pi/2 - j\sin 5\pi/2.$$

$$= 1 - 0.707 + j0.707 - j$$

For k=6

$$X(6) = \sum_{n=0}^{7} x(n)\, e^{-j3\pi n/2}$$

$$= x(0) + x(1)e^{-j3\pi/2} + x(2)e^{-j3\pi}$$

$$= 1 + \cos 3\pi/2 - j\sin 3\pi/2 + \cos 3\pi - j\sin 3\pi$$

$$= 1 + j - 1 = j$$

For k=7

$$X(7) = \sum_{n=0}^{7} x(n)\, e^{-j7\pi n/4}$$

$$= 1 + e^{-j7\pi/4} + e^{-j7\pi/2}$$

$$= 1 + \cos 7\pi/4 - j\sin 7\pi/4 + \cos 7\pi/2 - j\sin 7\pi/2$$

$$= 1 + 0.707 + j0.707 + j$$

$$= 1.707 + j1.707.$$

$$X(K) = \{3,\ 1.707 - j1.707,\ -j,\ 0.293 + j0.293,\ 1,\ 0.293 - j0.293,\ j,\ 1.707 + j1.707\}.$$

3) Find IDFT of the Sequence

$$X(K) = \{5, 0, 1-j, 0, 1, 0, 1+j, 0\}$$

Solution

IDFT: $x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{j2\pi kn/N}$, $n = 0, 1, \ldots N-1$

For N=8

$$x(n) = \frac{1}{8} \sum_{k=0}^{7} X(k)\, e^{-j\pi kn/4}, \quad n = 0, 1, \ldots N-1$$

For n=0

$$x(0) = \frac{1}{8} \sum_{k=0}^{7} X(k)\, e^{0}$$

$$= \frac{1}{8}[5 + 0 + 1 - j + 0 + 1 + 0 + 1 + j + 0]$$

$$= 1$$

For $n=1$

$$x(1) = \frac{1}{8}\left[\sum_{k=0}^{7} X(k) e^{j\pi k/4}\right]$$

$$= \frac{1}{8}\left[5 + (1-j)j + 1(-1) + (1+j)(-j)\right]$$

$$= \frac{1}{8}[6] = 0.75$$

For $n=2$

$$x(2) = \frac{1}{8}\left[\sum_{k=0}^{7} X(k) e^{j\pi k/2}\right]$$

$$= \frac{1}{8}\left[5 + (1-j)(-1) + 1(1) + (1+j)(-1)\right]$$

$$= \frac{1}{8}[4] = 0.5$$

For $n=3$

$$x(3) = \frac{1}{8}\left[\sum_{k=0}^{7} X(k) e^{j3\pi k/4}\right]$$

$$= \frac{1}{8}\left[5 + (1-j)(-j) + 1(-1) + (1+j)(j)\right]$$

$$= \frac{1}{8}(2) = 0.25$$

For $n=4$

$$x(4) = \frac{1}{8}\left[\sum_{k=0}^{7} X(k) e^{j\pi k}\right]$$

$$= \frac{1}{8}\left[5 + (1-j)(1) + 1(1) + (1+j)(1)\right]$$

$$= 1$$

For $n=5$

$$x(5) = \frac{1}{8}\left[\sum_{k=0}^{7} X(k) e^{j5\pi k/4}\right]$$

$$= \frac{1}{8}\left[5 + (1-j)(j) + (1)(-1) + (1+j)(-j)\right]$$

$$= \frac{1}{8}(6) = 0.75$$

For $n=6$

$$x(6) = \frac{1}{8}\left[\sum_{k=0}^{7} X(k) e^{j3\pi k/2}\right]$$

$$= \frac{1}{8}\left[5 + (1-j)(-1) + 1(1) + (1+j)(-1)\right]$$

$$= \frac{1}{8}(4) = 0.5$$

For $n=7$

$$x(7) = \frac{1}{8}\left[\sum_{k=0}^{7} X(k) e^{j7\pi k/4}\right] = 0.25$$

CIRCULAR CONVOLUTION BASED ON DFT AND IDFT METHOD

1) Perform the circular convolution of the following sequences $x(n) = \{1, 1, 2, 1\}$ & $h(n) = \{1, 2, 3, 4\}$ using DFT and IDFT method.

Solution:

By equation of circular convolution, we have

$$Y(k) = X(k)\,H(k).$$

$$X(k) = \sum_{n=0}^{N-1} x(n)\, e^{-j2\pi nk/N}, \quad k = 0, 1, \ldots N-1.$$

Here $N = 4$. & $x(n) = \{1, 1, 2, 1\}$ $\quad k = 0, 1, 2, 3$

For $k=0$.

$$X(0) = \sum_{n=0}^{3} x(n)\, e^{0}.$$

$$= x(0) + x(1) + x(2) + x(3)$$

$$= 5$$

For, $k=1$

$$X(1) = \sum_{n=0}^{3} x(n)\, e^{-j2\pi n/4}.$$

$$= x(0) + x(1)\, e^{-\pi/2} + x(2)\, e^{-\pi} + x(3)\, e^{-j3\pi/2}.$$

$$= 1 - j - 2 + j = -1.$$

For $k=2$

$$X(2) = \sum_{n=0}^{3} x(n)\, e^{-j2\pi n}$$

$$= x(0) + x(1)\, e^{-j\pi} + x(2)\, e^{-j2\pi} + x(3)\, e^{-j3\pi}.$$

$$= 1 - 1 + 2 - 1 = 1.$$

For $k=3$

$$x(3) = \sum_{n=0}^{3} x(n)\, e^{-j3\pi n/2}$$

$$= x(0) + x(1)\, e^{-j3\pi/2} + x(2)\, e^{-j3\pi} + x(3)\, e^{-j9\pi/2}.$$

$$= 1 + j - 2 - j = -1.$$

$$H(k) = \sum_{n=0}^{N-1} h(n) e^{-j2\pi nk/N} \quad , k = 0, 1, \ldots N-1$$

$h(n) = \{1, 2, 3, 4\}$.

For $k=0$

$$H(0) = \sum_{n=0}^{3} h(n)$$

$$= h(0) + h(1) + h(2) + h(3)$$

$$= 10$$

For $k=1$

$$H(1) = \sum_{n=0}^{3} h(n) e^{-j2\pi n/4}$$

$$= h(0) + h(1) e^{-j\pi/2} + h(2) e^{-j2\pi/2} + h(3) e^{-j3\pi/2}$$

$$= 1 - 2j - 3 + 4j$$

$$= -2 + j2$$

For $k=2$

$$H(2) = \sum_{n=0}^{3} h(n) e^{-j\pi n}$$

$$= h(0) e^{0} + h(1) e^{-j\pi} + h(2) e^{-j2\pi} + h(3) e^{-j3\pi}$$

$$= 1 + 2(-1) + 3(1) + 4(-1)$$

$$= -2$$

For $k=3$

$$H(3) = \sum_{n=0}^{3} h(n) e^{-j3\pi n/2}$$

$$= h(0) + h(1) e^{-j3\pi/2} + h(2) e^{-j3\pi} + h(3) e^{-j9\pi/2}$$

$$= 1 + 2j + 3(-1) + 4(-j)$$

$$= -2 - j2$$

$\therefore H(K) = \{10, -2+j2, -2, -2-j2\}$.

$y(K) = x(K) H(K) = \{50, 2-j2, -2, 2+j2\}$

By IDFT we have

$$y_3(n) = \frac{1}{N} \sum_{k=0}^{N-1} Y(k) e^{j2\pi nk/N}, \quad n = 0, 1, \ldots N-1.$$

For $n = 0$.

$$y(0) = \frac{1}{4} \sum_{k=0}^{3} Y(k) e^{j0}$$

$$= \frac{1}{4} \{ Y(0) + Y(1) + Y(2) + Y(3) \}$$

$$= \frac{1}{4} [50 + 2 - j2 - 2 + 2 + j2]$$

$$= 13$$

For $n = 1$

$$y(1) = \frac{1}{4} \sum_{k=0}^{3} Y(k) e^{j\pi k/2}$$

$$= \frac{1}{4} \left[Y(0) + Y(1) e^{j\pi/2} + Y(2) e^{j\pi} + Y(3) e^{j3\pi/2} \right]$$

$$= \frac{1}{4} [50 + (2 - j2) j + (-2)(-1) + (2 + j2)(-j)] = 14.$$

For $n = 2$

$$y(2) = \frac{1}{4} \sum_{k=0}^{3} Y(k) e^{j\pi k}$$

$$= \frac{1}{4} \left[Y(0) + Y(1) e^{j\pi} + Y(2) e^{j2\pi} + Y(3) e^{j3\pi} \right]$$

$$= \frac{1}{4} [50 + (2 - j2)(-1) + (-2) + (2 + j2)(-1)] = 11$$

For $n = 3$

$$y(3) = \frac{1}{4} \sum_{k=0}^{3} Y(k) e^{j3\pi k/2}.$$

$$= \frac{1}{4} \left[Y(0) + Y(1) e^{j3\pi/2} + Y(2) e^{+j3\pi} + Y(3) e^{j9\pi/2} \right]$$

$$= \frac{1}{4} [50 + (2 - j2)(-j) + (-2)(-1) + (2 + j2)(1)] = 12.$$

$$\therefore y(n) = \{ 13, 14, 11, 12 \}.$$

FILTERING METHODS BASED ON DFT:

When the input sequence is of long duration and has to be processed in a system whose impulse response is of finite duration, then it would not be able to store the data before convolution.

Therefore, the input sequence must be divided into blocks.

Each individual blocks are processed separately one at a time and the results are combined to yield the desired output sequence which is identical to the sequence obtained by linear convolution.

Two methods that are commonly used for filtering the sectioning data and combining the results are the overlap-save method and the overlap-add method.

i) OVERLAP SAVE METHOD

* Length of an i/p sequence be L_s.
* Length of an impulse response is M
* Input sequence is divided into blocks of data of size $N = L + M - 1$.
* Each block consists of last (M-1) data points of previous block followed by L new data points to form a data sequence of length $N = L + M - 1$.
* For first block of data the first M-1 points are set to zero.

$x_1(n) = \{ \underbrace{0, 0, 0 .. 0}_{(M-1) \text{ zeros}}, x(0), x(1), \ldots x(L-1) \}$

$x_2(n) = \{ \underbrace{x(L-M+1), \ldots, x(L-1)}_{\text{Last } (M-1) \text{ data points from } x_1(n)}, \underbrace{x(L), \ldots, x(2L-1)}_{\text{new data points}} \}$

Now the impulse response of the FIR filter is increased in length by appending L-1 zeros and an N-point circular convolution of $x_1(n)$ with $h(n)$ is computed. $y_1(n) = x_1(n) \circledN h(n)$.

In the output block $y_1(n)$, first M-1 points are corrupted and must be discarded.

ii) OVERLAP-ADD METHOD

* L_s & M be the length of the i/p sequence & impulse response

* The sequence is divided into blocks of data size having length L and M-1 zeros are appended to it to make the data size of L+M-1.

$x_1(n) = \{ x(0), x(1), \ldots x(L-1), \underbrace{0, 0}_{M-1 \text{ zeros}} \}$

$x_2(n) = \{ x(L), x(L+1), \ldots x(2L-1), \underbrace{0, 0}_{M-1 \text{ zeros}} \}$

* Now L-1 zeros are added to the impulse response h(n) and N-point circular convolution is performed.

* Since each data block is terminated with M-1 zeros, the last M-1 points from each output block must be overlapped and added to the first M-1 points of the succeeding block.

Let the o/p blocks are of the form

$y_1(n) = \{ y_1(0), y_1(1), \ldots, y_1(L-1), y_1(L), \ldots, y_1(N-1) \}$

$y_2(n) = \{ y_2(0), y_2(1), \ldots, y_2(L-1), y_2(L), \ldots, y_2(N-1) \}$

The output sequence is

$y(n) = \{ y_1(0), y_1(1), \ldots, y_1(L-1), y_1(L) + y_2(0), \ldots, y_1(N-1) + y_2(M-2), y_2(M-1) \}$

SECTIONED CONVOLUTION [Filtering Long duration Sequences].

1) Determine the output of the linear FIR filter whose impulse response $h(n) = \{1, 2, 3\}$ and the input signal $x(n) = \{1,2,3,4,5,6,7,8,9\}$ using overlap add method.

Solution:

Given $x(n) = \{1,2,3,4,5,6,7,8,9\} \leftarrow L_S$ (length of i/p sequence)

$h(n) = \{1, 2, 3\} \leftarrow M$ (Length of impulse)

$\therefore$ Length of o/p sequence $y(n) = L_S + M - 1$

Subdivide the i/p data sequence as 3 because the length of the impulse response $h(n)$.

To create a new length of each sequence,

$N = L + M - 1$

$= 3 + 3 - 1$

$N = 5$

To make the length as 5, add $(M-1)$ zeroes at each end of the sequence.

$x_1(n) = \{1, 2, 3, 0, 0\}$

$x_2(n) = \{4, 5, 6, 0, 0\}$

$x_3(n) = \{7, 8, 9, 0, 0\}$.

Now, we have i/p data sequence with length 5. So we must make the length of $h(n)$ as 5 by appending M-1 zeroes.

So $h(n) = \{1, 2, 3, 0, 0\}$.

$y_1(n) = x_1(n) * h(n)$

$$= \begin{bmatrix} 1 & 0 & 0 & 3 & 2 \\ 2 & 1 & 0 & 0 & 3 \\ 3 & 2 & 1 & 0 & 0 \\ 0 & 3 & 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 10 \\ 12 \end{bmatrix}$$

$y_2(n) = x_2(n) * h(n)$

$$= \begin{bmatrix} 4 & 0 & 0 & 6 & 5 \\ 5 & 4 & 0 & 0 & 6 \\ 6 & 5 & 4 & 0 & 0 \\ 0 & 6 & 5 & 4 & 0 \\ 0 & 0 & 6 & 5 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 13 \\ 28 \\ 27 \\ 18 \end{bmatrix}$$

$y_3(n) = x_3(n) * h(n)$

$$= \begin{bmatrix} 7 & 0 & 0 & 9 & 8 \\ 8 & 7 & 0 & 0 & 9 \\ 9 & 8 & 7 & 0 & 0 \\ 0 & 9 & 8 & 7 & 0 \\ 0 & 0 & 9 & 8 & 7 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 7 \\ 22 \\ 46 \\ 42 \\ 27 \end{bmatrix}$$

$y_1(n) \rightarrow$ 1 4 10 12 9
+ +
$y_2(n) \rightarrow$ 4 13 28 27 18
+ +
$y_3(n) \rightarrow$ 7 22 46 42 27

$y(n) = \{1, 4, 10, 16, 22, 28, 34, 40, 46, 42, 27\}$

2) Determine the output of the linear FIR filter whose impulse response is $h(n) = \{1, 1, 1\}$ and input of $x(n) = \{3, -1, 0, 3, 2, 0, 1, 2, 1\}$

Solution:

Given $x(n) = \{3, -1, 0, 1, 3, 2, 0, 1, 2, 1\}$.

$L_S = 10$.

$h(n) = \{1, 1, 1\}$ $M = 3$.

Length of $y(n) = L + M - 1 = 12$.

$N = L + M - 1 = 3 + 3 - 1 = 5$.

$h(n) = \{1, 1, 1, 0, 0\}$

$x_1(n) = \{0, 0, 3, -1, 0\}$

$x_2(n) = \{-1, 0, 1, 3, 2\}$

$x_3(n) = \{3, 2, 0, 1, 2\}$

$x_4(n) = \{1, 2, 1, 0, 0\}$

$y_1(n) = x_1(n) * h(n)$

$$y_1(n) = \begin{bmatrix} 0 & 0 & -1 & 3 & 0 \\ 0 & 0 & 0 & -1 & 3 \\ 3 & 0 & 0 & 0 & -1 \\ -1 & 3 & 0 & 0 & 0 \\ 0 & -1 & 3 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 3 \\ 2 \\ 2 \end{bmatrix}$$

$y_2(n) = x_2(n) * h(n)$

$$= \begin{bmatrix} -1 & 2 & 3 & 1 & 0 \\ 0 & -1 & 2 & 3 & 1 \\ 1 & 0 & -1 & 2 & 3 \\ 3 & 1 & 1 & -1 & 2 \\ 2 & 3 & 3 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 0 \\ 4 \\ 6 \end{bmatrix}$$

$y_3(n) = x_3(n) * h(n)$

$$= \begin{bmatrix} 3 & 2 & 1 & 0 & 2 \\ 2 & 3 & 2 & 1 & 0 \\ 0 & 2 & 3 & 2 & 1 \\ 1 & 0 & 2 & 3 & 2 \\ 2 & 1 & 0 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ 7 \\ 5 \\ 3 \\ 3 \end{bmatrix}$$

$y_4(n) = x_4(n) * h(n)$

$$= \begin{bmatrix} 1 & 0 & 0 & 1 & 2 \\ 2 & 1 & 0 & 0 & 1 \\ 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 4 \\ 3 \\ 1 \end{bmatrix}$$

$y_1(n) = \{-1, 0, 3, 2, 2\}$.

$y_2(n) = \{4, 1, 0, 4, 6\}$

$y_3(n) = \{6, 7, 5, 3, 3\}$

$y_4(n) = \{1, 3, 4, 3, 1\}$

$\therefore y(n) = \{3, 2, 2, 0, 4, 6, 5, 3, 3, 4, 3, 1\}$.

H.W

Find the o/p $y(n)$ of a filter whose impulse response is $h(n) = \{1, 1, 1\}$ & i/p signal $x(n) = \{3, -1, 0, 1, 3, 2, 0, 1, 2, -1\}$ using Overlap save & overlap add method.

H.W:

For the $x_1(n)$, $x_2(n)$ and N given Compute $x_1(n)$ Ⓝ $x_2(n)$

$x_1(n) = \delta(n) + \delta(n-1) + \delta(n-2)$,

$x_2(n) = 2\delta(n) - \delta(n-1) + 2\delta(n-2)$ ~~N = 4~~. N = 3.

FAST FOURIER TRANSFORM ALGORITHM (FFT)

The Fast Fourier Transform (FFT) is a highly efficient procedure for computing the DFT of a finite series and requires less number of computations than that of direct evaluation of DFT.

FFT computation technique is used in digital spectral analysis, filter simulation, auto correlation and pattern recognition.

FFT improves the performance by a factor 100 or more over direct evaluation of the DFT.

No. of complex multiplication in DFT is N^2

No. of complex multiplication in FFT is $\frac{N}{2}\log_2 N$.

DECIMATION IN TIME ALGORITHM

This algorithm is also known as Radix-2 DIT FFT algorithm.

The No. of output points $N = 2^M$, where M is an integer.

Let $x(n)$ is an N-point sequence, Decimate this sequence into two sequence of length $N/2$, where one sequence consisting of the even-indexed values of $x(n)$ and the other of odd indexed values of $x(n)$.

i.e> $x_e(n) = x(2n)$ $\quad n = 0, 1, \ldots, \frac{N}{2}-1$

$x_o(n) = x(2n+1)$ $\quad n = 0, 1, \ldots, \frac{N}{2}-1$

The N-point DFT of $x(n)$ can be written as

$$X(K) = \sum_{n=0}^{N-1} x(n) W_N^{nk} \quad K = 0, 1, \ldots N-1$$

Separating $x(n)$ into even and odd indexed values of $x(n)$, we obtain

$$X(K) = \sum_{\substack{n=0 \\ (\text{even})}}^{N-1} x(n) W_N^{nk} + \sum_{\substack{n=0 \\ (\text{odd})}}^{N-1} x(n) W_N^{nk}$$

$$= \sum_{n=0}^{N-1} x(2n) W_N^{2nk} + \sum_{n=0}^{N/2-1} x(2n+1) W_N^{(2n+1)k}$$

$$= \sum_{n=0}^{N/2-1} x(2n) W_N^{2nk} + W_N^{k} \sum_{n=0}^{N/2-1} x(2n+1) W_N^{2nk}$$

The above equation is in terms of even and odd indexed values of $x(n)$.

$$X(K) = \sum_{n=0}^{N/2-1} x_e(n) W_N^{2nk} + W_N^{k} \sum_{n=0}^{N/2-1} x_o(n) W_N^{2nk}$$

We can write

$$W_N^2 = \left[e^{-j2\pi/N} \right]^2 = e^{-j2\pi/N/2} = W_{N/2}$$

i.e> $W_N^2 = W_{N/2}$

By substituting this expression in $X(K)$, we get

$$X(K) = \underbrace{\sum_{n=0}^{N/2-1} x_e(n) W_{N/2}^{nk}}_{N/2\text{-point DFT of even indexed sequences}} + W_N^{K} \underbrace{\sum_{n=0}^{N/2-1} x_o(n) W_{N/2}^{nk}}_{N/2\text{-point DFT of odd indexed sequence}}.$$

$$X(K) = X_e(K) + W_N^{K} X_o(K) \qquad K = 0, 1, \ldots N/2 - 1.$$

$X_e(K)$ and $X_o(K)$ are periodic in k with period $N/2$.

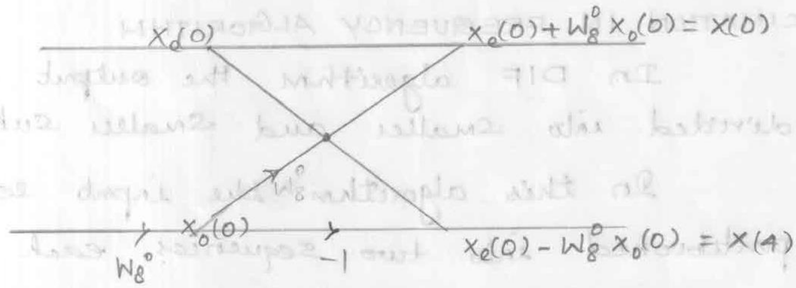
For $k \geq N/2$

$$W_N^{K+N/2} = -W_N^{K}$$

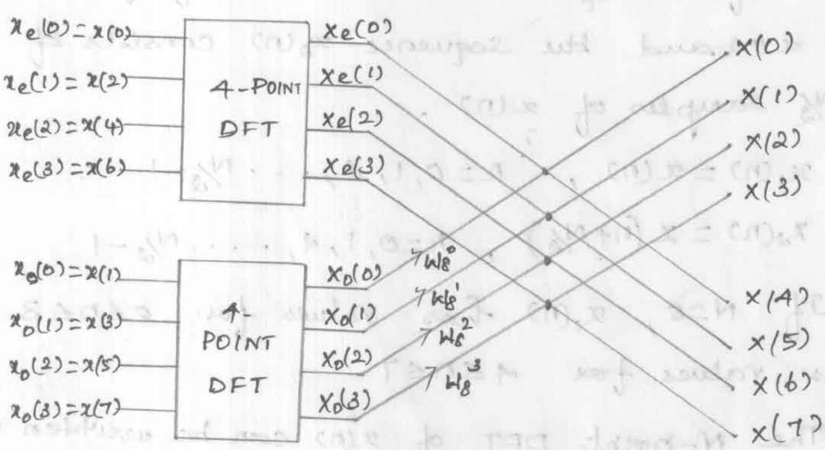
$$X(K) = X_e(K - N/2) - W_N^{K-N/2} X_o(K - N/2)$$

for $k = \frac{N}{2}, \frac{N}{2}+1, \ldots N-1$

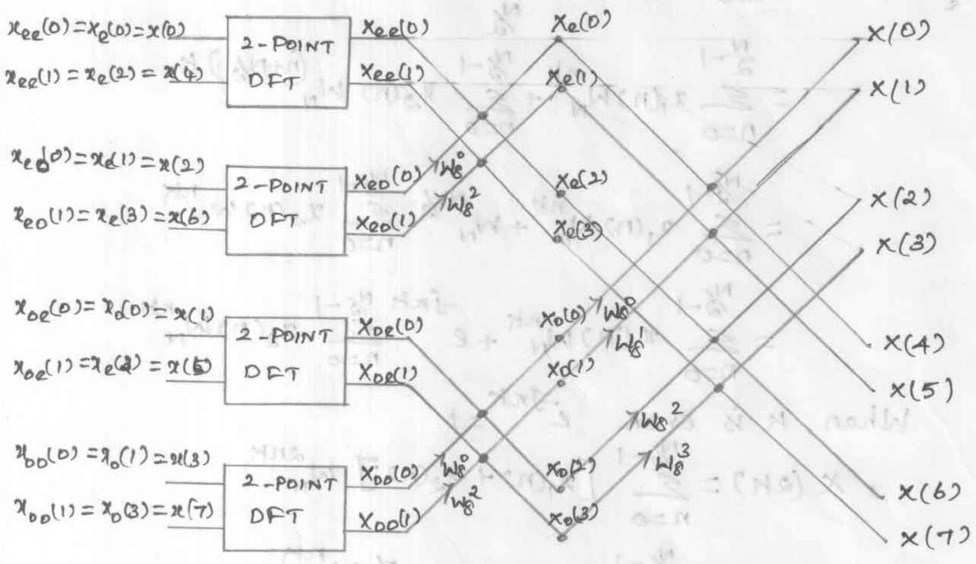
N = 8



FLOW GRAPH OF BUTTERFLY DIAGRAM.



CONSTRUCTION OF AN 8-POINT DFT FROM TWO 4 POINT DFTS.



CONSTRUCTION OF 8-POINT DFT FROM TWO 4-POINT DFTs AND 4-POINT DFT FROM TWO 2-POINT DFTs

DECIMATION IN FREQUENCY ALGORITHM

In DIF algorithm the output sequence $X(k)$ is divided into smaller and smaller subsequences.

In this algorithm the input sequence $x(n)$ is partitioned into two sequences each of length $N/2$ samples.

The first sequence $x_1(n)$ consist of first $N/2$ samples of $x(n)$ and the sequence $x_2(n)$ consists of the last $N/2$ samples of $x(n)$.

$$x_1(n) = x(n), \quad n = 0, 1, 2, \ldots N/2 - 1$$

$$x_2(n) = x(n + N/2), \quad n = 0, 1, 2, \ldots N/2 - 1$$

If $N = 8$, $x_1(n)$ has values for $0 \le n \le 3$ and $x_2(n)$ has values for $4 \le n \le 7$.

The N-point DFT of $x(n)$ can be written as

$$X(k) = \sum_{n=0}^{N/2-1} x(n) W_N^{nk} + \sum_{N/2}^{N-1} x(n) W_N^{nk}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} x_1(n) W_N^{nk} + \sum_{n=0}^{N/2-1} x_2(n) W_N^{(n+N/2)k}$$

$$= \sum_{n=0}^{N/2-1} x_1(n) W_N^{nk} + W_N^{Nk/2} \sum_{n=0}^{N/2-1} x_2(n) W_N^{nk}$$

$$= \sum_{n=0}^{N/2-1} x_1(n) W_N^{nk} + e^{-j\pi k} \sum_{n=0}^{N/2-1} x_2(n) W_N^{nk}$$

When k is even $e^{-j\pi k} = 1$

$$X(2k) = \sum_{n=0}^{N/2-1} [x_1(n) + x_2(n)] W_N^{2nk}$$

$$= \sum_{n=0}^{N/2-1} [x_1(n) + x_2(n)] W_{N/2}^{nk}$$

$$X(2k) = \sum_{n=0}^{N/2-1} f(n)\, W_{N/2}^{nk}$$

Where $f(n) = x_1(n) + x_2(n)$

The above eqn is the $N/2$ point DFT of the $N/2$ point sequence $f(n)$ obtained by adding the first half and the last half of the input sequence.

When k is odd

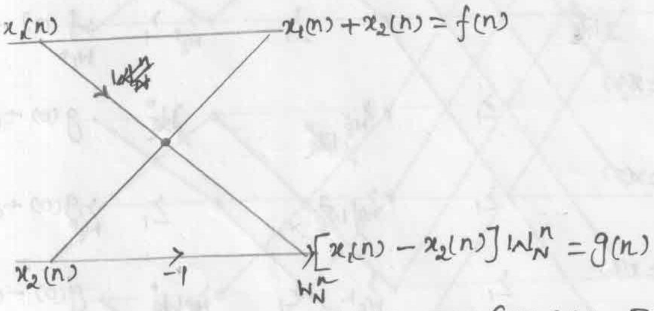
$$e^{-j\pi k} = -1$$

$$X(2k+1) = \sum_{n=0}^{\frac{N}{2}-1} [x_1(n) - x_2(n)]\, W_N^{(2k+1)n}$$

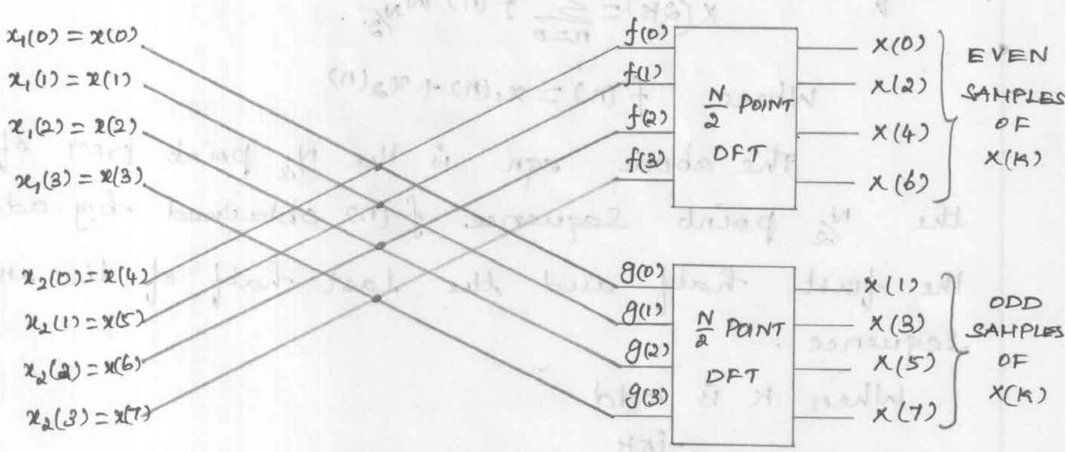
$$= \sum_{n=0}^{N/2-1} [x_1(n) - x_2(n)]\, W_N^{n}\, W_{N/2}^{nk}$$

$$= \sum_{n=0}^{N/2-1} g(n)\, W_{N/2}^{nk}$$

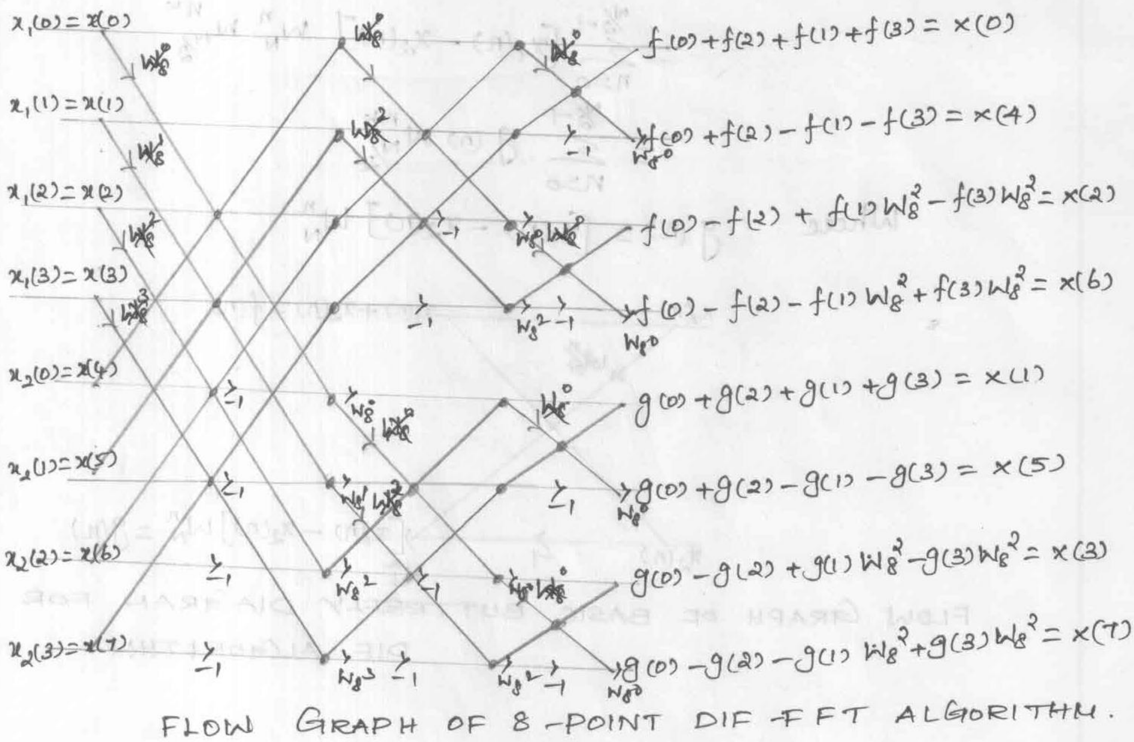
Where $g(n) = [x_1(n) - x_2(n)]\, W_N^{n}$



FLOW GRAPH OF BASIC BUTTERFLY DIAGRAM FOR DIF ALGORITHM



REDUCTION OF AN 8-POINT DFT TO TWO 4-POINT DFTs BY DECIMATION IN FREQUENCY.



FLOW GRAPH OF 8-POINT DIF-FFT ALGORITHM.

IDFT USING FFT ALGORITHM

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) W_N^{-kn}$$

$$N\, x^*(n) = \left[\sum_{k=0}^{N-1} X(k) W_N^{-kn} \right]^*$$

$$N\, x^*(n) = \sum_{k=0}^{N-1} X^*(k) W_N^{nk}$$

$$x(n) = \frac{1}{N} \left[\sum_{k=0}^{N-1} X^*(k) W_N^{nk} \right]^*$$

Hence IDFT can be found by taking Complex conjugate and then dividing the sequence by N.

USE OF FFT IN LINEAR FILTERING

Overlap add and Overlap save methods are used in conjunction with the FFT algorithm for Computing the DFT and the IDFT.

In this, the N-point DFT of h(n), which is padded by L-1 zeros, is denoted as H(k)

This computation is performed once via the FFT and the resulting N Complex numbers are stored. DIF-FFT algorithm is used to compute H(k).

$$Y_m(k) = H(k)\, X_m(k)$$

$$X_m(k) \rightarrow FFT\{x(n)\}.$$

$Y_m(k)$ in bit reversed order.

The IDFT can be computed by use of an FFT that takes the input in bit-reversed order and produces an output in normal Order.

1) Find the DFT of a sequence $x(n) = \{1, 2, 3, 4, 4, 3, 2, 1\}$ using DIT algorithm and DIF Algorithm.

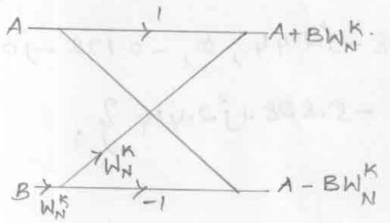
Solution

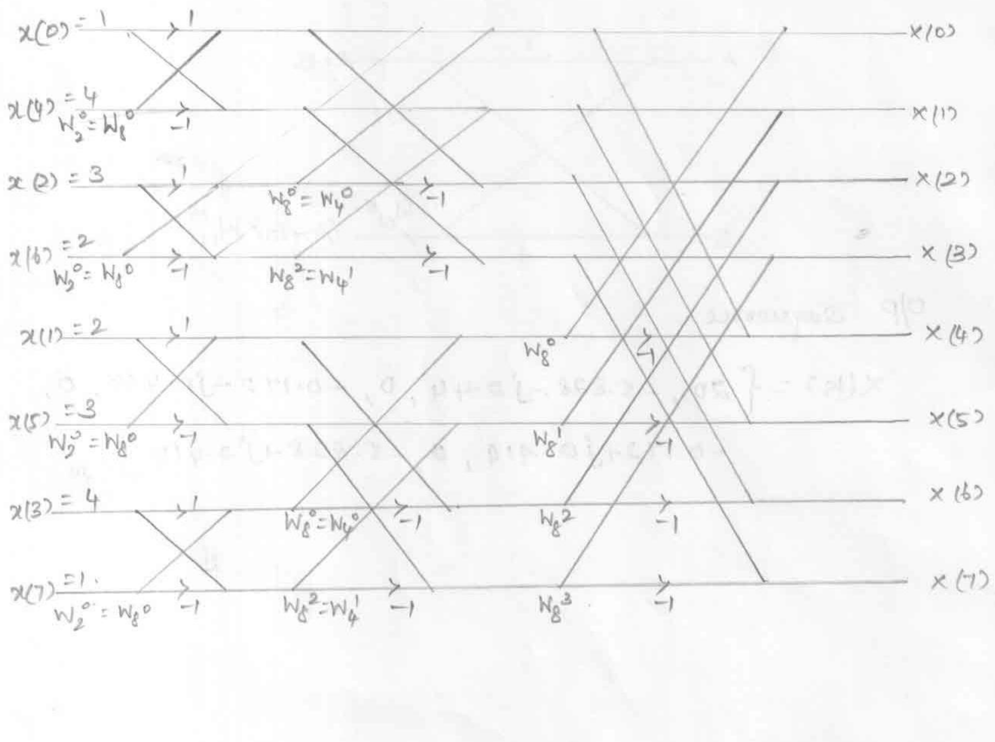
The twiddle factors associated with the flow graph are

$W_8^0 = 1$; $W_8^1 = (e^{-j2\pi/8})^1 = e^{-j\pi/4} = 0.707 - j0.707$

$W_8^2 = (e^{-j2\pi/8})^2 = e^{-j\pi/2} = -j$

$W_8^3 = (e^{-j2\pi/8})^3 = e^{-j3\pi/4} = -0.707 - j0.707$

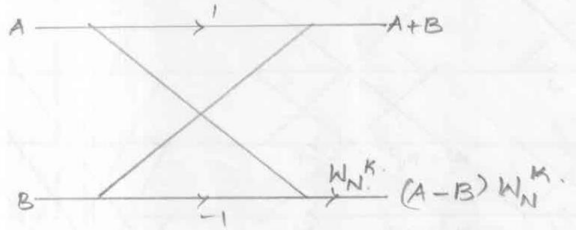




i/p	o/p of stage 1	o/p of stage 2	output
1	1+4=5	5+5=10	10+10=20
4	1−4=−3	−3+(−j)1=−3−j	−3−j+(0.707−j0.707)(−1−3j) = −5.828−j2.414
3	3+2=5	5−5=0	0
2	3−2=1	−3−(−j)=3+j	−3+j+(−0.707−j0.707)(−1+3j) = −0.172−j0.414
2	2+3=5	5+5=10	10−10=0
3	2−3=−1	−1+(−j)3=−1−3j	−3−j−(0.707−j0.707)(−1−3j) = −0.172+j0.414
4	4+1=5	5−5=0	0
1	4−1=3	−1−(−j)3=−1+3j	−3+j−(−0.707−j0.707)(−1+3j) = −5.828+j2.414

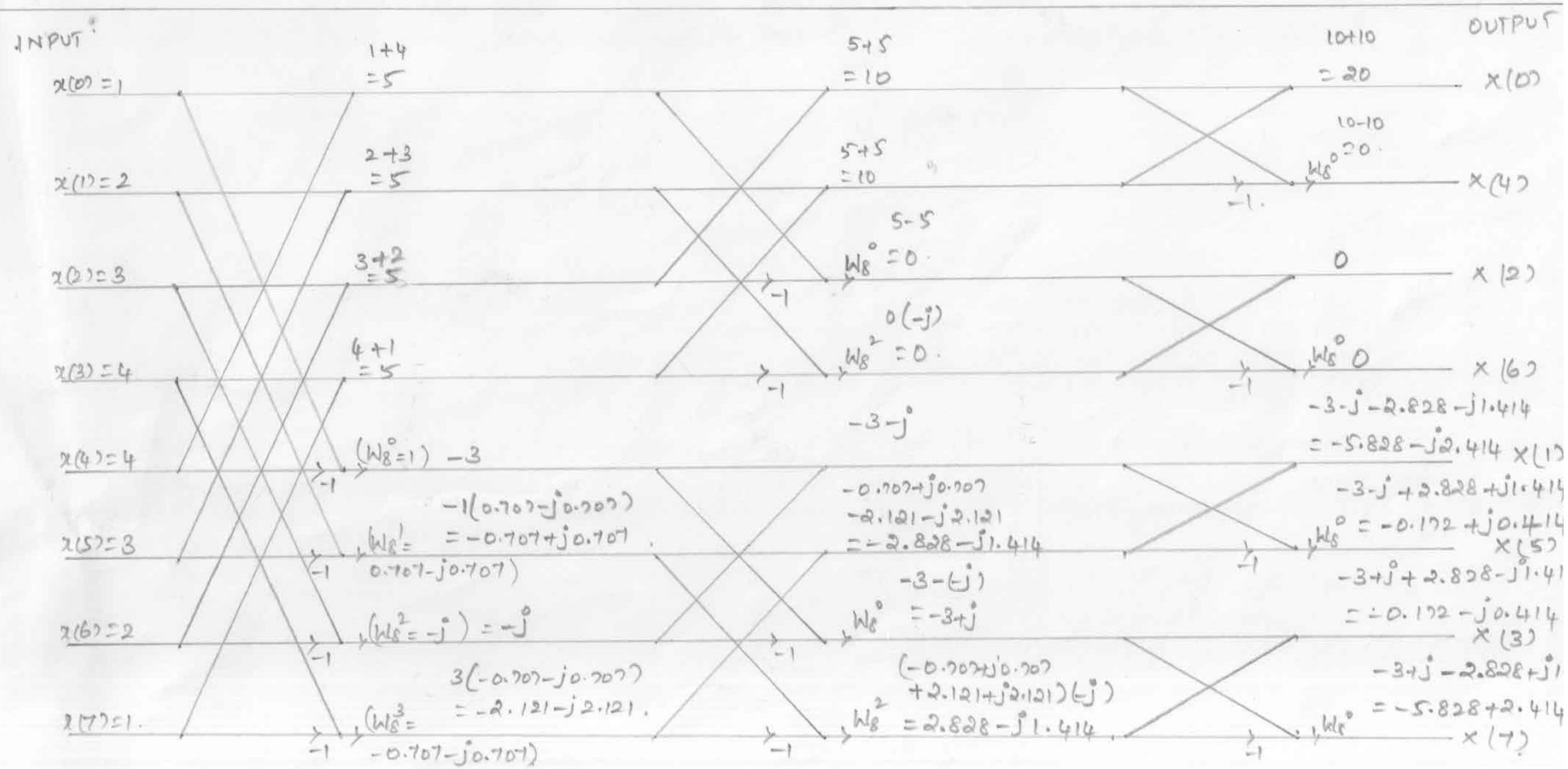
$X(K) = \{20, -5.828-j2.44, 0, -0.172-j0.414, 0, -0.172+j0.414, 0, -5.828+j2.414\}$.

DIF ALGORITHM.



O/P sequence

$X(K) = \{20, -5.828-j2.44, 0, -0.172-j0.414, 0, -0.172+j0.414, 0, -5.828+j2.414\}$


INPUT
x(0) = 1
x(1) = 2
x(2) = 3
x(3) = 4
x(4) = 4
x(5) = 3
x(6) = 2
x(7) = 1.
1+4 = 5
2+3 = 5
3+2 = 5
4+1 = 5
(W8^0 = 1) -3
-1(0.707-j0.707)
(W8^1 = 0.707-j0.707) = -0.707+j0.707
(W8^2 = -j) = -j
3(-0.707-j0.707)
(W8^3 = -0.707-j0.707) = -2.121-j2.121.
-1
5+5 = 10
5+5 = 10
5-5
W8^0 = 0.
0(-j)
W8^2 = 0.
-3-j
-0.707+j0.707
-2.121-j2.121
= -2.828-j1.414
-3-(-j)
W8^0 = -3+j
(-0.707+j0.707
+2.121+j2.121)(-j)
W8^2 = 2.828-j1.414
10+10 = 20
10-10
W8^0 = 0.
0
W8^0 0
-3-j-2.828-j1.414
= -5.828-j2.414
-3-j+2.828+j1.414
W8^0 = -0.172+j0.414
-3+j+2.828-j1.41
= -0.172-j0.414
-3+j-2.828+j1
W8^0 = -5.828+2.414
OUTPUT
X(0)
X(4)
X(2)
X(6)
X(1)
X(5)
X(3)
X(7)

UNIT-4

IIR FILTER DESIGN

STRUCTURES OF IIR (INTRODUCTION)

Infinite Impulse Response (IIR) are recursive type where present output samples depends on the present input, past input and past output samples.

A filter is one which rejects the unwanted frequencies from i/p signal and allow the desired frequency to obtain the o/p signal.

Passband:

The range of frequencies that are passed through the filter are called pass-band.

Ripples:

The limits of tolerance in the magnitude of passband and stopband are called ripples.

$\delta_p \rightarrow$ tolerance in passband

$\delta_s \rightarrow$ tolerance in stopband

STRUCTURES OF IIR FILTERS

System function of a recursive IIR filter is given by,

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}} \quad \text{——} (1)$$

DIRECT FORM I STRUCTURES

$$H_1(z) = \sum_{k=0}^{M} b_k z^{-k} \quad \text{[Numerator of eqn (1)]}$$

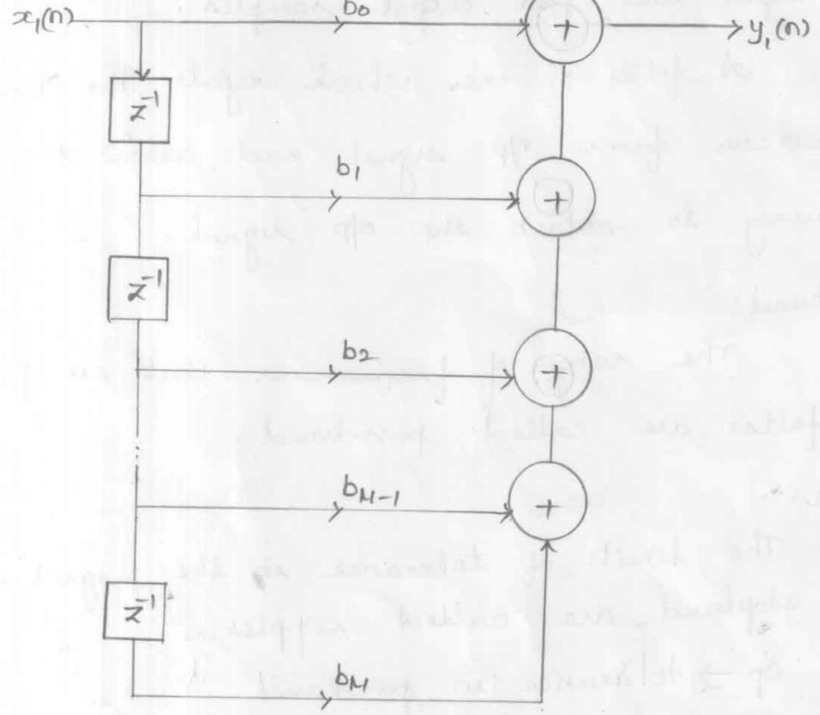
$$H_1(z) = b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_M z^{-M}$$

Since $H_1(z) = \dfrac{Y_1(z)}{X_1(z)}$,

$$Y_1(z) = b_0 X(z) + b_1 z^{-1} X_1(z) + b_2 z^{-2} X_1(z) + \cdots + b_M z^{-M} X_1(z)$$

Taking inverse z-transform of above equation

$$y_1(n) = b_0 x_1(n) + b_1 x_1(n-1) + b_2 x_1(n-2) + \cdots + b_M x_1(n-M)$$



DIRECT FORM REALIZATION OF SYSTEM FUNCTION $H_1(z)$

$$H_2(z) = \frac{1}{1 + \sum_{k=1}^{N} a_k z^{-k}}$$

$$\frac{Y_2(z)}{X_2(z)} = \frac{1}{1 + \sum_{k=1}^{N} a_k z^{-k}}$$

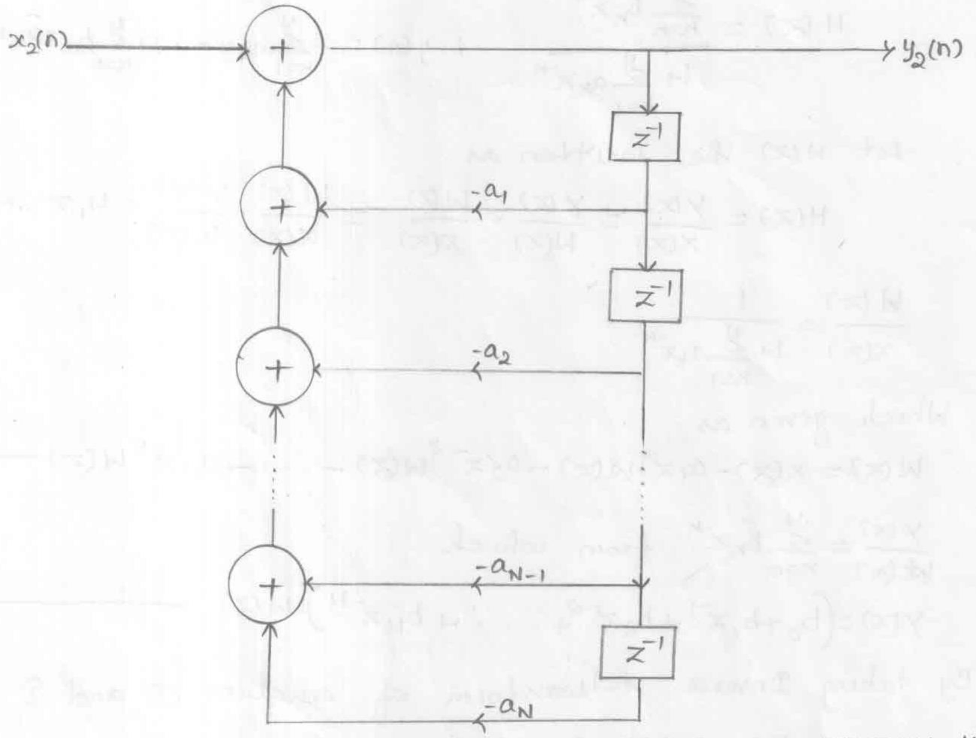
$$Y_2(z)\left[1 + \sum_{k=1}^{N} a_k z^{-k}\right] = X_2(z)$$

$$Y_2(z) = -\sum_{k=1}^{N} a_k z^{-k} Y_2(z) + X_2(z)$$

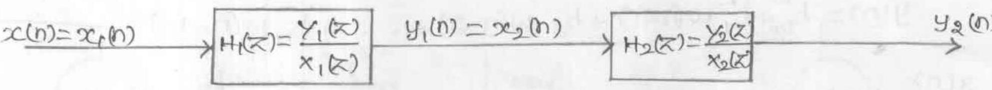
$$Y_2(z) = -a_1 z^{-1} Y_2(z) - a_2 z^{-2} Y_2(z) - \cdots - a_N z^{-N} Y_2(z) + X_2(z)$$

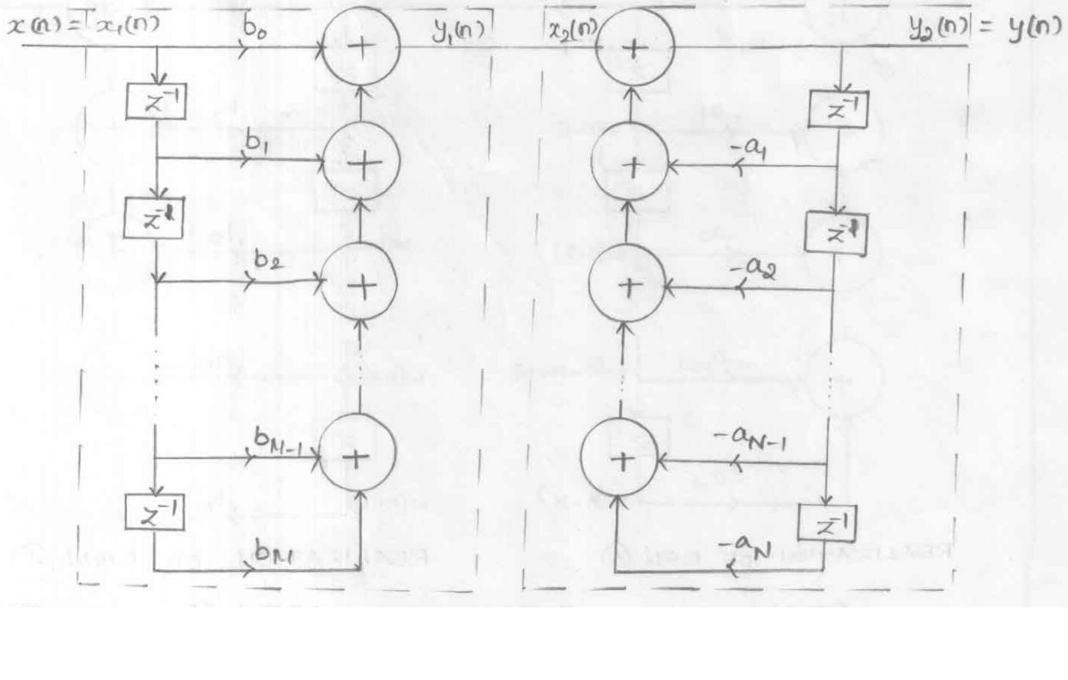
Taking inverse z-transform of above equation

$$y_2(n) = -a y_2(n-1) - a_2 y_2(n-2) - \dots a_N y_2(n-N) + x_2(n)$$



DIRECT FORM REALIZATION OF SYSTEM FUNCTION $H_2(z)$





DIRECT FORM-II STRUCTURE FOR IIR SYSTEM

System function of (LTI system) IIR filter is given by

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1+\sum_{k=1}^{N} a_k z^{-k}} \quad ; \quad y(n) = -\sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{M} b_k x(n-k) \rightarrow (A)$$

Let $H(z)$ be written as

$$H(z) = \frac{Y(z)}{X(z)} = \frac{Y(z)}{W(z)} \cdot \frac{W(z)}{X(z)} = \frac{W(z)}{X(z)} \cdot \frac{Y(z)}{W(z)} = H_1(z) \cdot H_2(z)$$

$$\frac{W(z)}{X(z)} = \frac{1}{1+\sum_{k=1}^{N} a_k z^{-k}}$$

Which given as

$$W(z) = X(z) - a_1 z^{-1} W(z) - a_2 z^{-2} W(z) - \dots - a_N z^{-N} W(z) \longrightarrow (1)$$

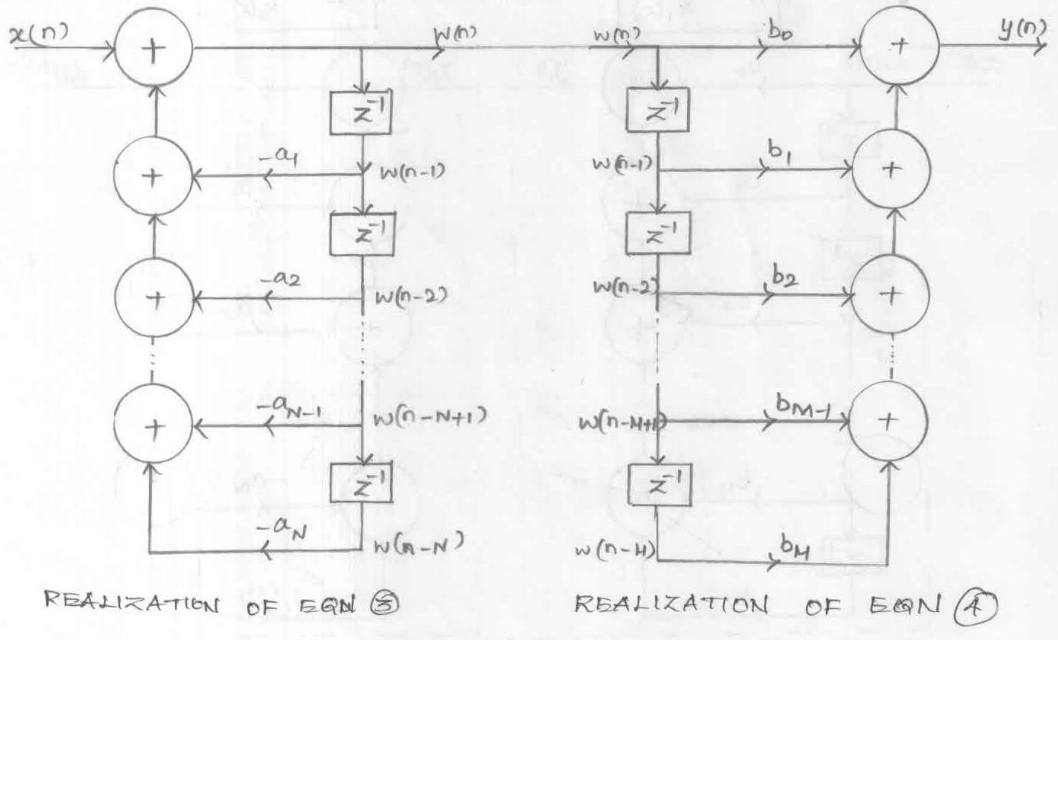
$$\frac{Y(z)}{W(z)} = \sum_{k=0}^{M} b_k z^{-k} \quad \text{from which}$$

$$Y(z) = \left(b_0 + b_1 z^{-1} + b_2 z^{-2} + \dots + b_M z^{-M}\right) W(z) \longrightarrow (2)$$

By taking Inverse z-transform of equations (1) and (2)

$$w(n) = x(n) - a_1 w(n-1) - a_2 w(n-2) - \dots - a_N w(n-N) \longrightarrow (3)$$

$$y(n) = b_0 w(n) + b_1 w(n-1) + b_2 w(n-2) + \dots + b_M w(n-M) \longrightarrow (4)$$



REALIZATION OF EQN (3)   REALIZATION OF EQN (4)

ANALOG FILTER DESIGN

DESIGN OF ANALOG LOWPASS BUTTERWORTH FILTER

The most general form of analog filter transfer function is

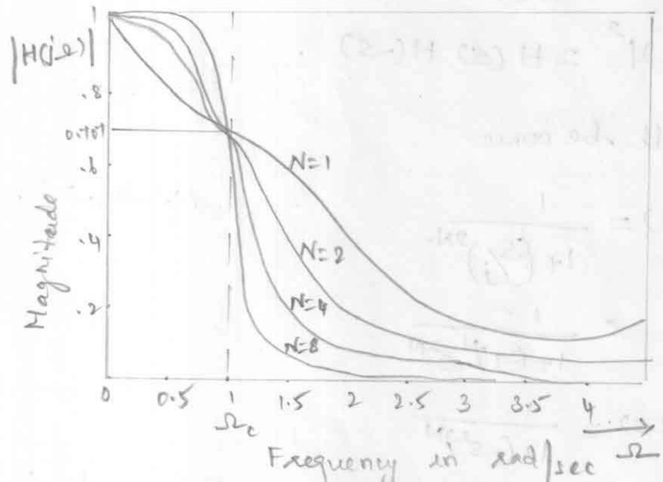
$$H(s) = \frac{N(s)}{D(s)} = \frac{\sum_{i=0}^{M} a_i s^i}{1+\sum_{i=1}^{N} b_i s^i}$$

The Butterworth filter is a type of signal processing filter designed to have as flat frequency response as possible in the pass band so that it is also termed as maximally flat magnitude filter.

The magnitude function of the Butterworth lowpass filter is given by

$$|H(j\Omega)|^2 = \frac{1}{1+\left(\frac{\Omega}{\Omega_c}\right)^{2N}} \longrightarrow (1)$$

Where
$N \rightarrow$ Order of the filter
$\Omega_c \rightarrow$ Cut-off frequency.
$N = 1, 2, 3 \ldots$



MAGNITUDE RESPONSE OF BUTTERWORTH LOWPASS FILTER

From the magnitude response plot, the function is monotonically decreasing, where the maximum response is unity at $\Omega = 0$.

The ideal response is shown by the dashed line. The magnitude response approaches the ideal lowpass characteristics as the Order N increases.

For values $\Omega < \Omega_c$; $|H(j\Omega)| \approx 1$, for values $\Omega > \Omega_c$; $|H(j\Omega)|$ decreases rapidly.

At $\Omega = \Omega_c$, the curves passes through 0.707, which corresponds to -3dB point.

In Order to derive transfer function of a stable filter, substitute $S = j\Omega$: in eqn ①.

$$|H(j\Omega)|^2 = H(\Omega^2)$$

$$= H\left[\left(\frac{S}{j}\right)^2\right] \quad (or)$$

$$= H(-S^2)$$

$$|H(j\Omega)|^2 = H(S)\, H(-S).$$

$$|H(j\Omega)|^2 = |H(S)|^2 = H(S)\, H(-S).$$

$\therefore$ Eqn ① will become

$$H(S)\,H(-S) = \frac{1}{1+\left(S/j\right)^{2N}} \qquad \because \Omega_c = 1.$$

$$= \frac{1}{1+(-1)^N S^{2N}}$$

$$H(S)\,H(-S) = \frac{1}{1+(-S^2)^N}$$

From the above eqn, H(S) has roots in the LHP and H(-S) has the corresponding roots in the RHP.

We can obtain these roots by equating the denominator to zero.

1) Given the specification $\alpha_p = 1$ dB ; $\alpha_s = 30$ dB ; $\Omega_p = 200$ rad/sec $\Omega_s = 600$ rad/sec. Determine the order of the filter.

Solution

Order of the filter, $N \geq \frac{\log A}{\log 1/K}$

$$A = \frac{\lambda}{\varepsilon} = \left[\frac{10^{0.1\alpha_s} - 1}{10^{0.1\alpha_p} - 1}\right]^{0.5}$$

$$= \left[\frac{10^3 - 1}{10^{0.1} - 1}\right]^{0.5}$$

$$A = 62.115$$

$$K = \frac{\Omega_p}{\Omega_s} = \frac{200}{600} = \frac{1}{3}$$

$$\therefore N \geq \frac{\log 62.115}{\log 3} \geq 3.758$$

$$\therefore N = 4$$

2) Design an analog Butterworth filter that has a -2 dB Passband attenuation at a frequency of 20 rad/sec and atleast -10 dB stopband attenuation at 30 rad/sec.

solution:

Given $\alpha_p = 2$ dB ; $\alpha_s = 10$ dB

$\Omega_p = 20$ rad/sec ; $\Omega_s = 30$ rad/sec.

Order of the filter, $N \geq \dfrac{\log \sqrt{\dfrac{10^{0.1\alpha_s} - 1}{10^{0.1\alpha_p} - 1}}}{\log \dfrac{\Omega_s}{\Omega_p}}$

$$\geq \frac{\log\sqrt{\frac{10-1}{10^{0.2}-1}}}{\log\frac{30}{20}}$$

$$\geq 3.37.$$

Rounding off N to the next highest integer we get

$N = 4$.

The Normalized Lowpass Butterworth filter for N=4 can be found written as,

$$H(S) = \frac{1}{(S^2+0.76537S+1)(S^2+1.8477S+1)}$$

cut off frequency, $\Omega_c = \frac{\Omega p}{(10^{0.1\alpha p}-1)^{1/2N}}$ (While stop band specification at Ω_s is exceeded)

$\Omega_c = \frac{\Omega_s}{(10^{0.1\alpha_s}-1)^{1/2N}}$ (While the passband specification at Ωp is exceeded)

Here stop band specification at Ω_s is exceeded. so

$$\Omega_c = \frac{\Omega p}{(10^{0.1\alpha p}-1)^{1/2N}}$$

$$\Omega_c = \frac{20}{(10^{0.2}-1)^{1/8}} = 21.3868.$$

The transfer function for $\Omega_c = 21.3868$ can be obtained by Substituting

$$S \rightarrow \frac{S}{\Omega_c} = \frac{S}{21.3868} \text{ in } H(S)$$

res

$$H(S) = \frac{1}{\left[\left(\frac{S}{21.3868}\right)^2+0.76537\left(\frac{S}{21.3868}\right)+1\right]\left[\left(\frac{S}{21.3868}\right)^2+1.8477\left(\frac{S}{21.3868}\right)+1\right]}$$

0.20921×10^6

STEPS TO DESIGN A DIGITAL FILTER USING IMPULSE INVARIANCE METHOD.

1) For the given specifications, find $H_a(S)$.

2) Select the sampling rate of the digital filter, T seconds per sample.

3) Express the analog filter transfer function as the sum of single-pole filters

$$H_a(S) = \sum_{K=1}^{N} \frac{C_K}{S - P_K}$$

4) Compute the z-transform of the digital filter by using the formula

$$H(Z) = \sum_{K=1}^{N} \frac{C_K}{1 - e^{P_K T} Z^{-1}}$$

For high sampling rate

$$H(Z) = \sum_{K=1}^{N} \frac{T\, C_K}{1 - e^{P_K T} Z^{-1}}$$

1) For the analog transfer function $H(s) = \frac{2}{(S+2)(S+2)}$ determine $H(z)$ using impulse invariance method. Assume $T = 1$ sec.

Solution:

Given $H(s) = \frac{2}{(S+1)(S+2)}$

Using partial fraction

$$H(s) = \frac{A}{S+1} + \frac{B}{S+2}$$

$$2 = A(S+2) + B(S+1)$$

Sub $S=-1$; $2 = A(1)$

$$A = 2$$

Sub $S=-2$; $2 = B(-1)$

$$B = -2$$

$$\therefore H(s) = \frac{2}{S+1} - \frac{2}{S+2}$$

$$H(s) = \frac{2}{S-(-1)} - \frac{2}{S-(-2)}$$

Using impulse invariance technique, if

$$H(s) = \sum_{K=1}^{N} \frac{C_K}{S-P_K} \quad \text{then} \quad H(z) = \sum_{K=1}^{N} \frac{C_K}{1-e^{P_K T} z^{-1}}.$$

ie) $(S-P_K)$ is transformed into $1-e^{P_K T} z^{-1}$.

There are two poles $P_1 = -1$ & $P_2 = -2$. So

$$H(z) = \frac{2}{1-e^{-T} z^{-1}} - \frac{2}{1-e^{-2T} z^{-1}}.$$

For $T = 1$ sec

$$H(z) = \frac{2}{1-e^{-1} z^{-1}} - \frac{2}{1-e^{-2} z^{-1}}$$

$$= \frac{2}{1-0.3678z^{-1}} - \frac{2}{1-0.1353z^{-1}}$$

$$H(z) = \frac{2(1-0.1353z^{-1}) - 2(1-0.3678z^{-1})}{\ldots} = \frac{0.465z^{-1}}{1-0.503z^{-1}+0.04976z^{-2}}$$

THE WARPING EFFECT

Let Ω and ω represent the frequency variables in the analog filter and the derived digital filter respectively.

$$\Omega = \frac{2}{T} \tan \frac{\omega}{2}.$$

For small value of ω

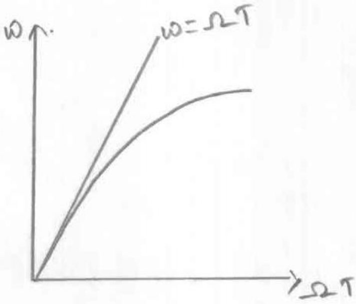
$$\Omega = \frac{2}{T} \cdot \frac{\omega}{2} = \frac{\omega}{T}$$

$$\omega = \Omega T$$

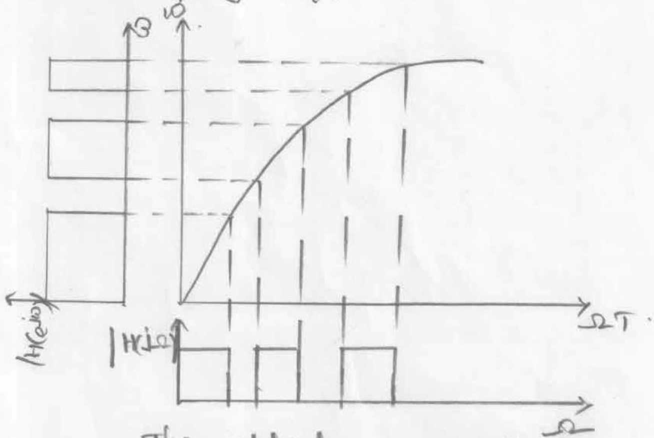
∵ For small value of θ; $\tan\theta = \theta$.

For low frequencies the relationship b/w Ω and ω are linear, as a result the digital filter have the same amplitude response as the analog filter.

For high frequencies, the relationship b/w Ω and ω becomes non-linear and distortion is introduced in the freq scale of the digital filter to that of the analog filter. This is known as the warping effect.



Relationship b/w Ω and ω



The effect on magnitude response due to warping effect

PREWARPING

The Warping effect can be eliminated by prewarping the analog filter. This can be done by finding prewarping analog frequencies using the formula

$$\Omega = \frac{2}{T}\tan\frac{\omega}{2}.$$

$$\therefore \Omega_p = \frac{2}{T}\tan\frac{\omega_p}{2} \quad \text{and}$$

$$\Omega_s = \frac{2}{T}\tan\frac{\omega_s}{2}$$

STEPS TO DESIGN DIGITAL FILTER USING BILINEAR TRANSFORM TECHNIQUE

1) From the given specifications, find prewarping analog frequencies using formula $\Omega = \frac{2}{T}\tan\frac{\omega}{2}$

2) Using the analog frequencies find $H(s)$ of the analog filter.

3) Select the sampling rate of the digital filter; call it T seconds per sample.

4) Substitute $s = \frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}$ into the transfer function found in step 2.

1) Apply bilinear transformation to $H(s) = \frac{2}{(s+1)(s+2)}$ with T = 1 sec and Find H(z).

Solution

Given $H(s) = \frac{2}{(s+1)(s+2)}$

Substitute $s = \frac{2}{T}\left[\frac{1-z^{-1}}{1+z^{-1}}\right]$ in H(s) to get H(z)

$$H(z) = H(s)\Big/_{s=\frac{2}{T}\left[\frac{1-z^{-1}}{1+z^{-1}}\right]}$$

$$H(z) = \frac{2}{(s+1)(s+2)}\Bigg/_{s=\frac{2}{T}\left[\frac{1-z^{-1}}{1+z^{-1}}\right]}$$

T = 1 sec.

$$H(z) = \frac{2}{\left\{2\left(\frac{1-z^{-1}}{1+z^{-1}}\right)+1\right\}\left\{2\left(\frac{1-z^{-1}}{1+z^{-1}}\right)+2\right\}}$$

$$= \frac{2}{\left\{\frac{2-2z^{-1}}{1+z^{-1}}+1\right\}\left\{\frac{2-2z^{-1}}{1+z^{-1}}+2\right\}}$$

$$= \frac{2(1+z^{-1})^2}{(2-2z^{-1}+1+z^{-1})(2-2z^{-1}+2+2z^{-1})}$$

$$= \frac{2(1+z^{-1})^2}{(3-z^{-1})\,4\,2}$$

$$= \frac{(1+z^{-1})^2}{6-2z^{-1}}$$

or

$$H(z) = \frac{0.166(1+z^{-1})^2}{(1-0.33z^{-1})}$$

FREQUENCY TRANSFORMATION IN ANALOG DOMAIN.

A system function of normalized filter is $H_a(s)$.

i) LOWPASS TO LOWPASS FILTER

Given LPF with a different cut-off frequency Ω_c or Passband frequency Ω_p

If we want another LPF with passband edge frequency of Ω_{LP}, the transformation is given by

$$s \rightarrow \frac{\Omega_p}{\Omega_{LP}} s \quad \left[s \rightarrow \frac{s}{\Omega_c} \right]$$

i.e.) Replace every 's' in $H_a(s)$ by $\frac{\Omega_p}{\Omega_{LP}} s$.

ii) LOWPASS TO HIGHPASS FILTER

Given a normalized LPF, it is desirable to have a highpass filter with cut-off frequency Ω_c. Then the transformation is given by

$$s \rightarrow \frac{\Omega_p \Omega_{HP}}{s} \quad \left[s \rightarrow \frac{\Omega_c}{s} \right]$$

iii) LOWPASS TO BANDPASS FILTER

The transformation for converting a normalized Low pass filter to a bandpass filter with cut off frequencies Ω_l, Ω_u can be accomplished by

$$s \rightarrow \frac{s^2 + \Omega_l \Omega_u}{s(\Omega_u - \Omega_l)}$$

iv) LOWPASS TO BANDSTOP FILTER (NOTCH FILTER)

The Transformation to convert a normalized lowpass filter to a bandstop filter is

$$s \rightarrow \frac{s(\Omega_u - \Omega_l)}{s^2 + \Omega_l \Omega_u}.$$

DESIGN OF HIGHPASS, BANDPASS AND BANDSTOP FILTERS.

To find the transfer function of highpass, bandpass and band stop filters of any type first find the transfer function of normalized LPF. ($H_a(s)$) and then use suitable transformation is used.

1) For the given specifications $\alpha_p = 3dB$; $\alpha_s = 15dB$, $\Omega_p = 1000$ rad/sec and $\Omega_s = 500$ rad/sec design a highpass filter.

Solution:

First we design a normalized LPF and then suitable transformation is used to get the transfer function of a highpass filter.

For LPF

$\Omega_c = \Omega_p = 500$ rad/sec

$\Omega_s = 1000$ rad/sec.

$\alpha_p = 3dB$, $\alpha_s = 15dB$

For HPF

$\Omega_c = \Omega_p = 1000$ rad/sec

$\Omega_s = 500$ rad/sec.

Order of the filter, $N = \dfrac{\log \lambda/\varepsilon}{\log\left(\frac{\Omega_s}{\Omega_p}\right)}$

$$\lambda = \sqrt{10^{0.1\alpha_s} - 1} = \sqrt{10^{0.1\times 15dB} - 1} = 5.533$$

$$\varepsilon = \sqrt{10^{0.1\alpha_p} - 1} = \sqrt{10^{0.1\times 3dB} - 1} = 1$$

$$\frac{\Omega_p}{\Omega_s} = 0.5$$

$$\therefore N = \frac{\log 5.533}{\log 2} = 2.468$$

$N = 3$ [By approximating Next higher integer

$H(s)$ for $\Omega_c = 1$ rad/sec and $N = 3$ is

$$H(s) = \frac{1}{(s+1)(s^2+s+1)}$$

To get highpass filter having cut-off frequency

$\Omega_c = \Omega_p = 1000$ rad/sec.

Sub $s \to \frac{1000}{s}$

$$H_a(s) = H(s)\Big|_{s \to \frac{1000}{s}}$$

$$= \frac{1}{(s+1)(s^2+s+1)}\Bigg|_{s \to \frac{1000}{s}}$$

$$= \frac{1}{\left(\frac{1000}{s}+1\right)\left[\frac{(1000)^2}{s^2} + \frac{1000}{s} + 1\right]}$$

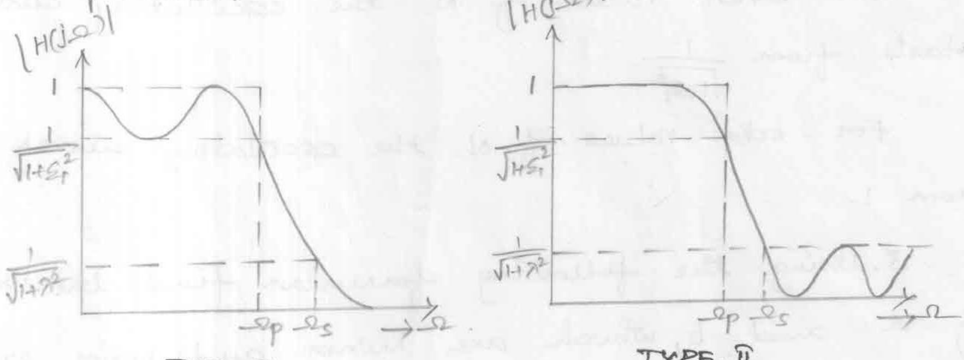
$$= \frac{1}{\left(\frac{1000+s}{s}\right)\left(\frac{(1000)^2 + 1000s + s^2}{s^2}\right)}$$

$$H_a(s) = \frac{s^3}{(s+1000)\ s^2+1000s+(1000)^2}$$

ANALOG LOWPASS CHEBYSHEV FILTERS

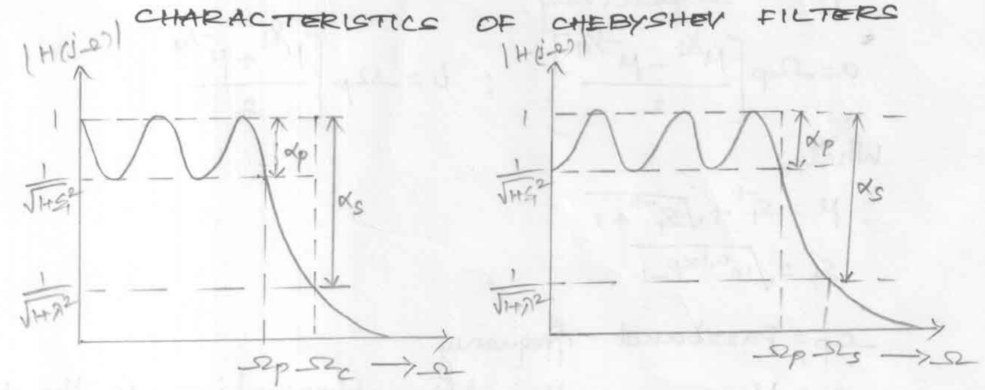
These are two types of chebyshev filters. Type I chebyshev filters are all-pole filters that exhibit equiripple behaviour in the passband and a monotonic characteristics in the stop band.

Type II chebyshev filter contains both poles and zeros and exhibits a monotonic behaviour in the passband and an equiripple behaviour in the stop band.



TYPE I TYPE II

CHARACTERISTICS OF CHEBYSHEV FILTERS



LOWPASS CHEBYSHEV FILTER MAGNITUDE RESPONSE

STEPS TO DESIGN AN ANALOG CHEBYSHEV LOWPASS FILT

1. From the given specifications find the order of the filter N using the formula

$$N \geq \left(\cosh^{-1}\sqrt{\frac{10^{0.1\alpha_s}-1}{10^{0.1\alpha_p}-1}}\right) \Big/ \cosh^{-1}\left(\frac{\Omega_s}{\Omega_p}\right)$$

Where $\cosh^{-1}x = \ln\left(x+\sqrt{x^2-1}\right) = \dfrac{\log\left(x+\sqrt{x^2-1}\right)}{\log 2}$

2. Round off it to the next higher integer.

3. For even values of N the oscillatory curves starts from $\dfrac{1}{\sqrt{1+\varepsilon_1^2}}$

For odd values of N the oscillatory curves starts from 1.

3. Using the following formulas find the values of a and b, which are minor and major axis of the ellipse respectively.

$$a = \Omega_p\left[\frac{\mu^{1/N}-\mu^{-1/N}}{2}\right] \;; \quad b = \Omega_p\left[\frac{\mu^{1/N}+\mu^{-1/N}}{2}\right]$$

Where,

$\mu = \varepsilon_1^{-1} + \sqrt{\varepsilon_1^{-2}+1}$

$\varepsilon_1 = \sqrt{10^{0.1\alpha_p}-1}$

Ω_p = Passband frequency

α_p = Maximum allowable attenuation in the passband

[∵ For normalized chebyshev filter $\Omega_p = 1$ rad/sec

4. Calculate the poles of chebyshev filter which lie on an ellipse by using the formula

$$S_k = a\cos\phi_k + jb\sin\phi_k \quad , k = 1, 2, \ldots N$$

Where $\phi_k = \pi/2 + \left(\frac{2k-1}{2N}\right)\pi \quad , \quad k = 1, 2, \ldots N$

5. Find the denominator polynomial of the transfer function using the above poles

i.e> $(S_k - a\cos\phi_k)^2 + (b\sin\phi_k)^2$

6. The numerator of the transfer function depends on the value of N.

a> For N odd substitute $s=0$ in the denominator polynomial and find the value. This value is equal to the numerator of the transfer function.

b> For N even substitute $s=0$ in the denominator polynomial and divide the result by $\sqrt{1+\epsilon_1^2}$. This value is equal to the numerator.

1. Given the specifications $\alpha_p = 3$ dB ; $\alpha_s = 16$ dB ; $f_p = 1$ KHz and $f_s = 2$ KHz. Determine the order of the filter using chebyshev approximation. Find H(s).

Solution

From the given data we can find

$$\Omega_p = 2\pi \times 1000\,\text{Hz} = 2000\pi \text{ rad/sec}$$

$$\Omega_s = 2\pi \times 2000\,\text{Hz} = 4000\pi \text{ rad/sec}$$

and $\alpha_p = 3$ dB ; $\alpha_s = 16$ dB

i> Order of the filter, N

$$N \geq \frac{\cosh^{-1}\sqrt{\frac{10^{0.1\alpha_s}-1}{10^{0.1\alpha_p}-1}}}{\cosh^{-1}\left(\frac{\Omega_s}{\Omega_p}\right)} \geq \frac{\cosh^{-1}\sqrt{\frac{10^{1.6}-1}{10^{0.3}-1}}}{\cosh^{-1}\left(\frac{4000\pi}{2000\pi}\right)}$$

$$\geq \frac{\cosh^{-1}(6.244)}{\cosh^{-1}(2)}$$

$$\geq \frac{\ln\left[6.244+\sqrt{(6.244)^2-1}\right]}{\ln\left[2+\sqrt{2^2-1}\right]}$$

$$\geq \frac{3.6332}{1.8999}$$

$$N \geq 1.91$$

2) Rounding N to next higher value i.e. $N=2$.

For N even, the oscillatory curve starts from $\frac{1}{\sqrt{1+\epsilon_1^2}}$

3) The values of minor axis and major axis can be found as

$$\epsilon_1 = \left(10^{0.1\alpha_p}-1\right)^{0.5} = \sqrt{10^{0.3}-1} = 1$$

$$\mu = \epsilon_1^{-1} + \sqrt{1+\epsilon_1^{-2}} = 1+\sqrt{2} = 2.414$$

$$a = \Omega_p\left[\frac{\mu^{1/N}-\mu^{-1/N}}{2}\right] = 2000\pi\left[\frac{(2.414)^{1/2}-(2.414)^{-1/2}}{2}\right]$$

$$a = \frac{2000\pi}{2}(0.91) = 910\pi.$$

$$b = \Omega_p\left[\frac{\mu^{1/N}+\mu^{-1/N}}{2}\right] = 2000\pi\left[\frac{(2.414)^{1/2}+(2.414)^{-1/2}}{2}\right]$$

$$b = 2197\pi.$$

4) The poles are given by

$$S_k = a\cos\phi_k + jb\sin\phi_k, \quad k=1,2$$

$$\phi_k = \frac{\pi}{2}+\frac{(2k-1)\pi}{2N}, \quad k=1,2$$

$$\phi_1 = \frac{\pi}{2}+\frac{\pi}{4} = 135^\circ$$

$s_1 = a\cos\phi_1 + jb\sin\phi_1 = -643.46\pi + j1554\pi$.

$s_2 = a\cos\phi_2 + jb\sin\phi_2 = -643.46\pi - j1554\pi$.

5). The Denominator of $H(s) = (s+643.46\pi)^2 + (1554\pi)^2$

6) The Numerator of $H(s) = \dfrac{(643.46\pi)^2 + (1554\pi)^2}{\sqrt{1+\epsilon_1^2}}$ [$\because$ N is even]

$= (1414.38)^2 \pi^2$

$\therefore$ The transfer function $H(s) = \dfrac{(1414.38)^2\pi^2}{(s+643.46\pi)^2 + (1554\pi)^2}$

$$H(s) = \frac{(1414.38)^2\pi^2}{s^2 + 1287\pi s + (1682)^2\pi^2}$$

UNIT-5

DIGITAL SIGNAL PROCESSORS

ARCHITECTURE

The ’54x DSPs use an advanced, modified Harvard architecture that maximizes processing power by maintaining one program memory bus and three data memory buses. These processors also provide an arithmetic logic unit (ALU) that has a high degree of parallelism, application-specific hardware logic, on-chip memory, and additional on-chip peripherals. These DSP families also provide a highly specialized instruction set, which is the basis of the operational flexibility and speed of these DSPs. Separate program and data spaces allow simultaneous access to program instructions and data, providing the high degree of parallelism. Two reads and one write operation can be performed in a single cycle. Instructions with parallel store and application-specific instructions can fully utilize this architecture. In addition, data can be transferred between data and program spaces. Such parallelism supports a powerful set of arithmetic, logic, and bit-manipulation operations that can all be performed in a single machine cycle. Also included are the control mechanisms to manage interrupts, repeated operations, and function calls. Figure 1 shows the functional block diagram that shows the principal blocks and bus structure in the ’54x devices

TMS320C5’416 FAMILY – FUNCTIONAL OVERVIEW

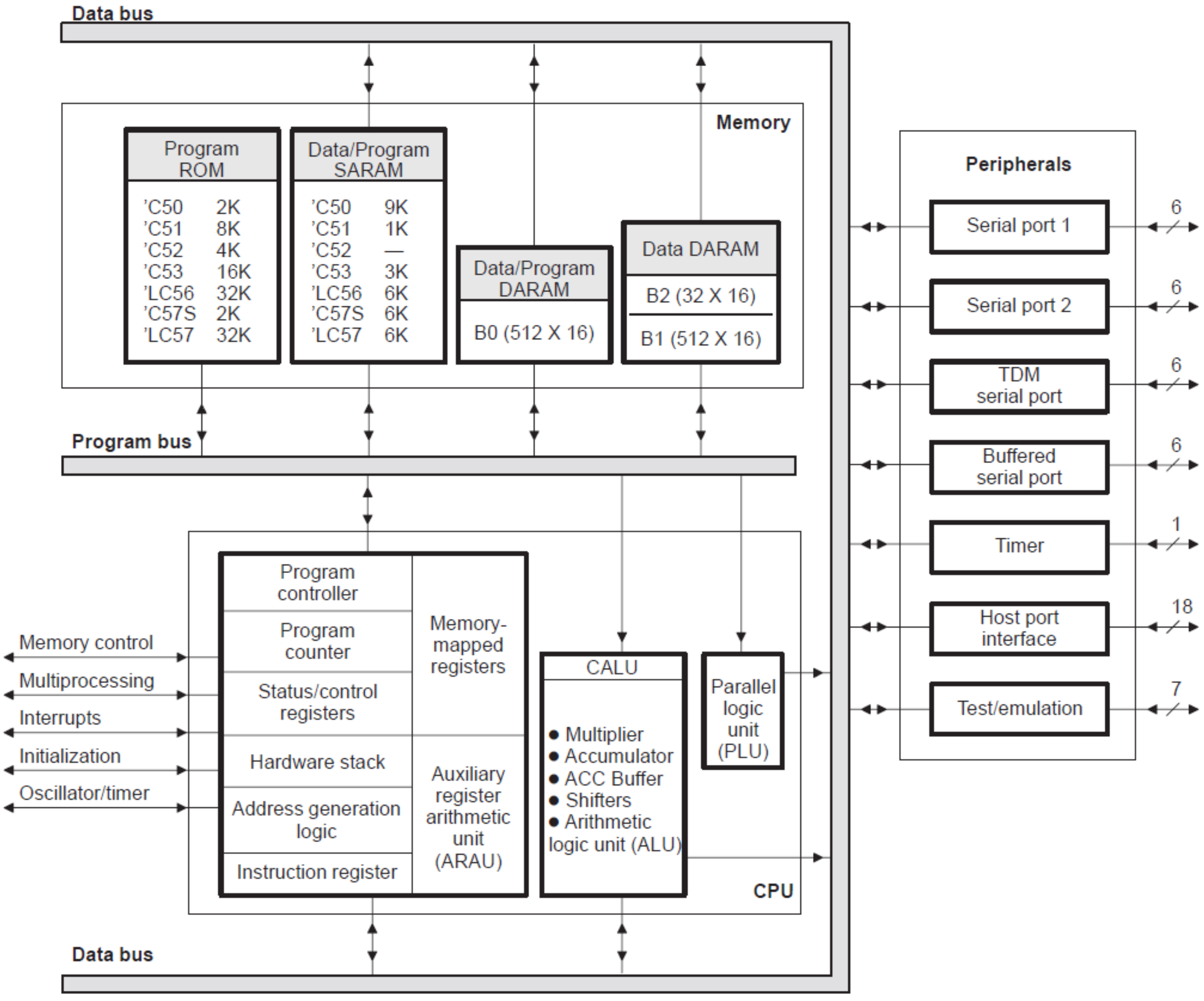
Central Processing Unit (CPU)

The CPU of the ’54x devices contains:

- A 40-bit arithmetic logic unit (ALU)
- Two 40-bit accumulators
- A barrel shifter
- A 17 × 17-bit multiplier/adder

Arithmetic Logic Unit (ALU)

The ’54x devices perform 2s-complement arithmetic using a 40-bit ALU and two 40-bit accumulators (ACCA and ACCB). The ALU also can perform Boolean operations. The ALU can function as two 16-bit ALUs and perform two 16-bit operations simultaneously when the C16 bit in status register 1 (ST1) is set.



Accumulators

The accumulators, ACCA and ACCB, store the output from the ALU or the multiplier / adder block; the accumulators can also provide a second input to the ALU or the multiplier / adder. The bits in each accumulator is grouped as follows:

 Guard bits (bits 32–39)

- A high-order word (bits 16–31)

- A low-order word (bits 0–15)

Instructions are provided for storing the guard bits, the high-order and the low-order accumulator words in data memory, and for manipulating 32-bit accumulator words in or out of data memory. Also, any of the accumulators can be used as temporary storage for the other.

Barrel Shifter

The '54x's barrel shifter has a 40-bit input connected to the accumulator or data memory (CB, DB) and a 40-bit output connected to the ALU or data memory (EB). The barrel shifter produces a left shift of 0 to 31 bits and a right shift of 0 to 16 bits on the input data. The shift requirements are defined in the shift-count field (ASM) of ST1 or defined in the temporary register (TREG), which is designated as a shift-count register. This shifter and the exponent detector normalize the values in an accumulator in a single cycle. The least significant bits (LSBs) of the output are filled with 0s and the most significant bits (MSBs) can be either zero-filled or sign-extended, depending on the state of the sign-extended mode bit (SXM) of ST1. Additional shift capabilities enable the processor to perform numerical scaling, bit extraction, extended arithmetic, and overflow prevention operations.

Multiplier/Adder

The multiplier / adder performs 17 × 17-bit 2s-complement multiplication with a 40-bit accumulation in a single instruction cycle. The multiplier / adder block consists of several elements: a multiplier, adder, signed/unsigned input control, fractional control, a zero detector, a rounder (2s-complement), overflow/saturation logic, and TREG. The multiplier has two inputs: one input is selected from the TREG, a data-memory operand, or an accumulator; the other is selected from the program memory, the data memory, an accumulator, or an immediate value. The fast on-chip multiplier allows the '54x to perform operations such as convolution, correlation, and filtering efficiently. In addition, the multiplier and ALU together execute multiply/accumulate (MAC) computations and ALU operations in parallel in a single instruction cycle. This function is used in determining the Euclid distance, and in implementing symmetrical and least mean square (LMS) filters, which are required for complex DSP algorithms.

Compare, Select, and Store Unit (CSSU)

The compare, select, and store unit (CSSU) performs maximum comparisons between the accumulator's high and low words, allows the test/control (TC) flag bit of status register 0 (ST0) and the transition (TRN) register to keep their transition histories, and selects the larger word in the accumulator to be stored in data memory. The CSSU also accelerates Viterbi-type butterfly computation with optimized on-chip hardware.

Program Control

Program control is provided by several hardware and software mechanisms:

- The program controller decodes instructions, manages the pipeline, stores the status of operations, and decodes conditional operations. Some of the hardware elements included in the program controller are the program counter, the status and control register, the stack, and the address-generation logic.

- Some of the software mechanisms used for program control include branches, calls, conditional instructions, a repeat instruction, reset, and interrupts.

- The '54x supports both the use of hardware and software interrupts for program control. Interrupt service routines are vectored through a relocatable interrupt vector table. Interrupts can be globally enabled/disabled and can be individually masked through the interrupt mask register (IMR). Pending interrupts are indicated in the interrupt flag register (IFR). For detailed information on the structure of the interrupt vector table, the IMR and the IFR, see the device-specific data sheets.

Status Registers (ST0, ST1)

The status registers, ST0 and ST1, contain the status of the various conditions and modes for the '54x devices. ST0 contains the flags (OV, C, and TC) producedarithmetic operations and bit manipulations in addition to the data page pointer (DP) and the auxiliary register pointer (ARP) fields. ST1 contains the various modes and instructions that the processor operates on and executes.

Auxiliary Registers (AR0–AR7)

The eight 16-bit auxiliary registers (AR0–AR7) can be accessed by the central arithmetic logic unit (CALU) and modified by the auxiliary register arithmetic units (ARAUs). The primary function of the auxiliary registers is generating 16-bit addresses for data space. However, these registers also can act as general-purpose registers or counters.

Temporary Register (TREG)

The TREG is used to hold one of the multiplicands for multiply and multiply/accumulate instructions. It can hold a dynamic (execution-time programmable) shift count for instructions with a shift operation such as ADD, LD, and SUB. It also can hold a dynamic bit address for the BITT instruction. The EXP instruction stores the exponent value computed into the TREG, while the NORM instruction uses the TREG value to normalize the number. For ACS operation of Viterbi decoding, TREG holds branch metrics used by the DADST and DSADT instructions.

Transition Register (TRN)

The TRN is a 16-bit register that is used to hold the transition decision for the path to new metrics to perform the Viterbi algorithm. The CMPS (compare, select, max, and store) instruction updates the contents of the TRN based on the comparison between the accumulator high word and the accumulator low word.

Stack-Pointer Register (SP)

The SP is a 16-bit register that contains the address at the top of the system stack. The SP always points to the last element pushed onto the stack. The stack is manipulated by interrupts, traps, calls, returns, and the PUSHD, PSHM, POPD, and POPM instructions. Pushes and pops of the stack predecrement and postincrement, respectively, all 16 bits of the SP.

Circular-Buffer-Size Register (BK)

The 16-bit BK is used by the ARAUs in circular addressing to specify the data block size.

Block-Repeat Registers (BRC, RSA, REA)

The block-repeat counter (BRC) is a 16-bit register used to specify the number of times a block of code is to be repeated when performing a block repeat. The block-repeat start address (RSA) is a 16-bit register containing the starting address of the block of program memory to be repeated when operating in the repeat mode. The 16-bit block-repeat end address (REA) contains the ending address if the block of program memory is to be repeated when operating in the repeat mode.

Interrupt Registers (IMR, IFR)

The interrupt-mask register (IMR) is used to mask off specific interrupts individually at required times. The interrupt-flag register (IFR) indicates the current status of the interrupts.

Processor-Mode Status Register (PMST)

The processor-mode status register (PMST) controls memory configurations of the '54x devices.

Power-Down Modes

There are three power-down modes, activated by the IDLE1, IDLE2, and IDLE3 instructions. In these modes, the ’54x devices enter a dormant state and dissipate considerably less power than in normal operation. The IDLE1 instruction is used to shut down the CPU. The IDLE2 instruction is used to shut down the CPU and on-chip peripherals. The IDLE3 instruction is used to shut down the ’54x processor completely. This instruction stops the PLL circuitry as well as the CPU and peripherals.

BUS STRUCTURE

The ’54x device architecture is built around eight major 16-bit buses:

 One program-read bus (PB) which carries the instruction code and immediate operands from program memory

 Two data-read buses (CB, DB) and one data-write bus (EB), which interconnect to various elements, such as the CPU, data-address generation logic (DAGEN), program-address generation logic (PAGEN), on-chip peripherals, and data memory

 The CB and DB carry the operands read from data memory.

- The EB carries the data to be written to memory.

 Four address buses (PAB, CAB, DAB, and EAB), which carry the addresses needed for instruction execution The ’54x devices have the capability to generate up to two data-memory addresses per cycle, which are stored into two auxiliary register arithmetic units (ARAU0 and ARAU1).

The PB can carry data operands stored in program space (for instance, a coefficient table) to the multiplier for multiply /accumulate operations or to a destination in data space for the data-move instruction. This capability allows implementation of single-cycle three-operand instructions such as FIRS. The ’54x devices also have an on-chip bidirectional bus for accessing on-chip peripherals; this bus is connected to DB and EB through the bus exchanger in the CPU interface. Accesses using this bus can require more than two cycles for reads and writes depending on the peripheral’s structure.

The ’54x devices can have bus holders connected to the data bus and the HPI data bus. Bus holders ensure that the data bus does not float. When bus holders are enabled, the data bus maintains its previous level. Setting bit 1 of the bank-switching control register (BSCR) enables bus holders and clearing bit 1 disables the bus holders. A reset automatically disables the bus holders. Selected devices also have equivalent bus holders connected to the address bus. The bus holders ensure that the address bus does not float when in high impedance. For these devices, the bus holders are always enabled.

MEMORY

The minimum memory address range for the ’54x devices is 192K words — composed of 64K words in program space, 64K words in data space, and 64K words in I/O space. Selected devices also provide extended program memory space of up to 8M words. The program memory space contains the instructions to be executed as well as tables used in execution. The data memory space stores data used by the instructions. The I/O memory space interfaces to external memory-mapped peripherals and can also serve as extra data storage space. The ’54x DSPs provide both on-chip RAM and ROM to improve system performance and integration.

On-Chip ROM

The ’54x devices include on-chip maskable ROM that can be mapped into program memory or data memory depending on the device. On-chip ROM is mapped into program space by the microprocessor/microcontroller (MP/MC) mode control pin. On-chip ROM that can be mapped into data space is controlled by the DROM bit in the processor mode status register (PMST). This allows an instruction to use data stored in the ROM as an operand. Customers can arrange to have the ROM of the ’54x programmed with contents unique to any particular application.

Bootloader

A bootloader is available in the standard ’54x on-chip ROM. This bootloader can be used to transfer user code from an external source to anywhere in the program memory at power up automatically. If the MP/MC pin of the device is sampled low during a hardware reset, execution begins at location FF80h of the on-chip ROM. This location contains a branch instruction to the start of the bootloader program.

On-Chip Dual-Access RAM (DARAM)

Dual-access RAM blocks can be accessed twice per machine cycle. This memory is intended primarily to store data values; however, it can be used to store program as well. At reset, the DARAM is mapped into data memory space. DARAM can be mapped into program/data memory space by setting
the OVLY bit in the PMST register.

On-Chip Single-Access RAM (SARAM)

Each of the SARAM blocks is a single-access memory. This memory is intended primarily to store data values; however, it can be used to store program as well. SARAM can be mapped into program/data memory space by setting the OVLY bit in the PMST register.

On-Chip Two-Way Shared RAM

Select 54x devices with multiple CPU cores include two-way shared RAM blocks that allow simultaneous program space access from two CPU cores. Each CPU can perform a single access with zero-states to any location in the two-way shared RAM during each clock cycle. This shared RAM is most efficiently used when the two CPUs are executingidentical programs. In this case, the amount of program memory required for the application is effectively reduced by 50% since both CPUs can execute from the same RAM.

On-Chip Memory Security

A security feature is included on 54x devices to prevent the on-chip memory contents from being extracted by a user. This feature is enabled during the manufacturing process and is ONLY available to customers that order custom ROM programming. Consequently, memory security cannot be enabled/disabled by the user.

Program Memory

The standard external program memory space on the ’54x devices addresses up to 64K 16-bit words. Software can configure their memory cells to reside inside or outside of the program address map. When the cells are mapped into program space, the device automatically accesses them when their addresses are within bounds. When the program-address generation (PAGEN) logic generates an address outside its bounds, the device automatically generates an external access.

PIPELINING

Instruction pipelining is a technique that implements a form of parallelism called instruction-level parallelism within a single processor. It therefore allows faster CPU throughput (the number of instructions that can be executed in a unit of time) than would otherwise be possible at a given clock rate. The basic instruction cycle is broken up into a series called a pipeline. Rather than processing each instruction sequentially (finishing one instruction before starting the next), each instruction is split up into a sequence of steps so different steps can be executed in parallel and instructions can be processed concurrently (starting one instruction before finishing the previous one). identical programs. In this case, the amount of program memory required for the application is effectively reduced by 50% since both CPUs can execute from the same RAM.

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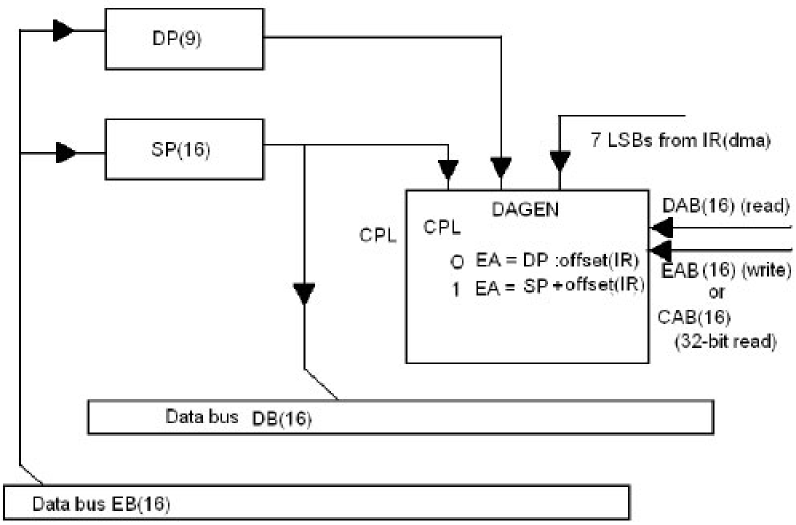


Indirect Addressing
• Data space is accessed by address present in an auxiliary register. • 54xx have 8, 16 bit auxiliary register (AR0 – AR 7). Two auxiliary register arithmetic units (ARAU0 & ARAU1) • Used to access memory location in fixed step size. AR0 register is used for indexed and bit reverse addressing modes. • For single – operand addressing MOD type of indirect addressing ARF AR used for addressing
• ARP depends on (CMPT) bit in ST1 CMPT = 0, Standard mode, ARP set to zero CMPT = 1, Compatibility mode, Particularly AR selected by ARP

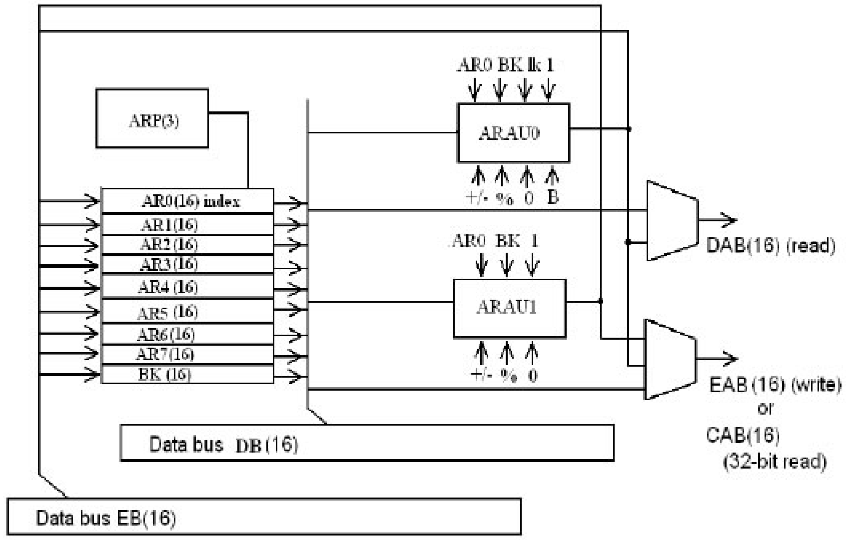


Table 1 Indiect addressing option with single data memory operand ; Circular addressing\

Operand syntax	Function
*ARx	Addr = ARx;
*ARx -	Addr = ARx ; ARx = ARx -1
*ARx +	Addr = ARx; ARx = ARx +1
*+ARx	Addr = ARx+1; ARx = ARx +1
*ARx - 0B	Addr = ARx ; ARx = B(ARx – AR0)
*ARx - 0	Addr = Arx ; ARx = ARx – AR0
*ARx + 0	Addr = Arx ; ARx = ARx +AR0
*ARx + 0B	Addr = ARx ; ARx = B(ARx + AR0)
*ARx - %	Addr = ARx ; ARx = circ(ARx – 1)

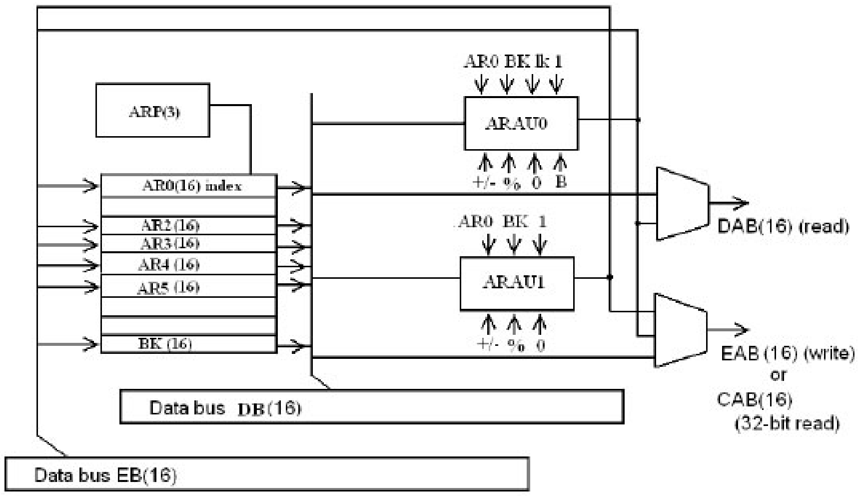
Bit-Reversed Addressing:

• Used for FFT algorithms. • AR0 specifies one half of the size of the FFT. • The value of AR0 = 2N-1: N = integer FFT size = 2N • AR0 + AR (selected register) = bit reverse addressing. • The carry bit propagating from left to right.

Dual-Operand Addressing

Dual data-memory operand addressing is used for instruction that simultaneously perform two reads (32-bit read) or a single read (16-bit read) and a parallel store (16-bit store) indicated by two vertical bars, II. These instructions access operands using indirect addressing mode.

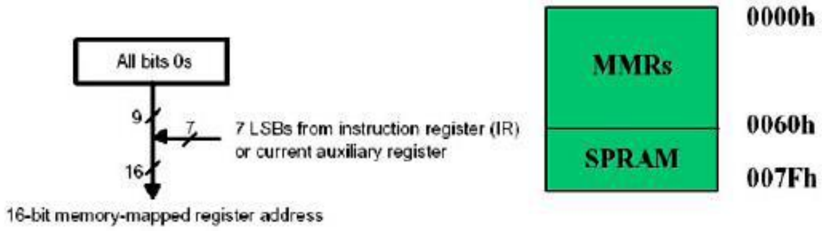
If in an instruction with a parallel store the source operand the destination operand point to the same location, the source is read before writing to the destination. Only 2 bits are available in the instruction code for selecting each auxiliary register in this mode. Thus, just four of the auxiliary registers, AR2-AR5, can be used, The ARAUs together with these registers, provide capability to access two operands in a single cycle. Figure 6 shows how an address is generated using dual data-memory operand addressing.



Memory-Mapped Register Addressing

• Used to modify the memory-mapped registers without affecting the current datapage pointer (DP) or stack-pointer (SP) – Overhead for writing to a register is minimal – Works for direct and indirect addressing – Scratch –pad RAM located on data PAGE0 can be modified • STM #x, DIRECT • STM #tbl, AR1

Figure



Stack Addressing

• Used to automatically store the program counter during interrupts and subroutines. • Can be used to store additional items of context or to pass data values. • Uses a 16-bit memory-mapped register, the stack pointer (SP). • PSHD X2